Advanced Calculus 2, Dr. Block, Lecture Notes, 4-10-2020

We begin discussing material from Section 7.3 of the text. Today we will look at two versions of the ratio test.

Theorem 25. (d'Alembert's Ratio Test) Let $\sum a_k$ be a series with all terms positive.

a. Suppose that there is a positive integer J and a real number $\alpha < 1$ such that $\frac{a_{n+1}}{a_n} \leq \alpha$ for all $n \geq J$. Then the series $\sum a_k$ converges.

b. Suppose that there is a positive integer J such that $\frac{a_{n+1}}{a_n} \geq 1$ for all $n \geq J$. Then the series $\sum a_k$ diverges.

Proof of Part a. Suppose that the hypothesis is satisfied. We observe the following:

$$a_{J+1} \leq \alpha \cdot a_J$$

$$a_{J+2} \leq \alpha \cdot a_{J+1} \leq (\alpha)^2 \cdot a_J$$

and for each positive integer n,

$$a_{J+n} \leq (\alpha)^n \cdot a_J.$$

Now, we know that the series $\sum(\alpha)^k$ converges (geometric series). So the series $\sum((\alpha)^k \cdot a_J)$ also converges. It now follows from the Comparison Test that the series $\sum a_k$ converges.

Proof of Part b. Suppose that the hypothesis is satisfied. Then $a_{n+1} \geq a_n$ for all $n \geq J$. It follows that $a_n \geq a_J$ for all $n \geq J$. Since Since a_J is a positive constant, it follows that the sequence $\{a_n\}$ does not converge to zero. Thus, the series $\sum a_k$ diverges, by the Test for Divergence.

□

We also have the following Corollary.

Corollary 26. (Cauchy's Ratio Test) Let $\sum a_k$ be a series with all terms positive. Suppose that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L$$

where L is a real number or ∞.

a. If $L < 1$, then the series $\sum a_k$ converges.

b. If $L > 1$, then the series $\sum a_k$ diverges.

Proof of Part a. Suppose that the hypothesis is satisfied. Set $\alpha = \frac{1+L}{2}$. Then $L < \alpha < 1$. It follows from the definition of a limit of a sequence that there is a positive integer J such that $\frac{a_{n+1}}{a_n} < \alpha$ for all $n \geq J$. So the series $\sum a_k$ converges, by Theorem 25 (Part a).

Proof of Part b. Suppose that the hypothesis is satisfied. It follows from the definition of a limit of a sequence that there is a positive integer J such that $\frac{a_{n+1}}{a_n} > 1$ for all $n \geq J$. So the series $\sum a_k$ diverges, by Theorem 25 (Part b).

□

Here is an example.

Problem 27. Determine whether the given series converges or diverges.

$$\sum \frac{(k!)^2}{(2k)!}$$

Solution We try to use one of the ratio tests. We have

$$\frac{a_{k+1}}{a_k} = \frac{((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{(k!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)}.$$

It follows that

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \frac{1}{4}.$$

So, Cauchy's Ratio Test (Part a) can be applied, and we conclude that the given series converges.

□

Next, we will go over Problem 16 from Section 7.1 of the text.

Problem 28. If a series $\sum |a_k|$ converges, and a sequence $\{b_n\}$ is bounded, prove that $\sum a_k b_k$ converges.

Proof. Suppose that the series $\sum |a_k|$ converges, and the sequence $\{b_n\}$ is bounded. We can assume without loss of generality that the series $\sum |a_k|$ and the sequence $\{b_n\}$ start at the integer 1. We will prove that the series $\sum |a_k b_k|$ converges. Let T_n denote the n-th term in the sequence of partial sums for the series $\sum |a_k|$. Then

$$T_n = |a_1| + \cdots + |a_n|.$$

Since the series $\sum |a_k|$ converges, the sequence $\{T_n\}$ is bounded. (See Theorem 15, from the Lecture notes 4-3-2020.) So there is some real number D such that $T_n \le D$ for each positive integer n. Also, as the sequence $\{b_n\}$ is bounded, there is some real number B such that

$$|b_n| \le B$$

for each positive integer n. Now, let S_n denote the n-th term in the sequence of partial sums for the series $\sum |a_k b_k|$. Then for each postive integer n we have

$$S_n = |a_1||b_1| + \cdots + |a_n||b_n| \le B \cdot T_n \le B \cdot D.$$

Hence, the sequence $\{S_n\}$ is bounded. It follows (again from Theorem 15) that the series $\sum |a_k b_k|$ converges. Finally, by Theorem 16 from the Lecture notes 4-3-2020, the series $\sum a_k b_k$ also converges.