Advanced Calculus 2, Dr. Block, Lecture Notes, 4-13-2020

We continue discussing material from Section 7.3 of the text. Please work on the following exercises: Page 315 - 317, Exercises 2, 4, 7, 10, 13, 14 (all parts of each).

Today we will look at two versions of the root test.

Theorem 29. (Root Test) Let $\sum a_k$ be a series with all terms positive.

a. Suppose that there is a positive integer J and a real number $\alpha < 1$ such that $\sqrt[n]{a_n} \leq \alpha$ for all $n \geq J$. Then the series $\sum a_k$ converges.

b. Suppose that there is a positive integer J such that $\sqrt[n]{a_n} \geq 1$ for all $n \geq J$. Then the series $\sum a_k$ diverges.

Proof of Part a. Suppose that the hypothesis is satisfied. Then for each integer $n \geq J$, we have $a_n \leq \alpha^n$. Now, we know that the series $\sum \alpha^k$ converges (geometric series). It follows from the Comparison Test that the series $\sum a_k$ converges.

Proof of Part b. Suppose that the hypothesis is satisfied. Then $a_n \geq 1$ for all $n \geq J$. It follows that the sequence $\{a_n\}$ does not converge to zero. Thus, the series $\sum a_k$ diverges, by the Test for Divergence.

□

We also have the following Corollary.

Corollary 30. (Cauchy's Root Test) Let $\sum a_k$ be a series with all terms positive. Suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L$$

where L is a real number or ∞.

a. If $L < 1$, then the series $\sum a_k$ converges.

b. If $L > 1$, then the series $\sum a_k$ diverges.

Proof of Part a. Suppose that the hypothesis is satisfied. Set $\alpha = \frac{1+L}{2}$. Then $L < \alpha < 1$. It follows from the definition of a limit of a sequence that there is a positive integer J such that $\sqrt[n]{a_n} < \alpha$ for all $n \geq J$. So the series $\sum a_k$ converges, by Theorem 29 (Part a).

Proof of Part b. Suppose that the hypothesis is satisfied. It follows from the definition of a limit of a sequence that there is a positive integer J such that $\sqrt[n]{a_n} > 1$ for all $n \geq J$. So the series $\sum a_k$ diverges, by Theorem 29 (Part b).

□

Here is an example. This is Exercise 13 on Page 317 of the text.

Problem 31. Determine whether the given series converges or diverges.

$$\sum a_k = \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^5} + \ldots$$

Solution. Observe that for the given series we have for each positive integer n,

$$a_n = \begin{cases} \dfrac{1}{3^n} & \text{if } n \text{ is odd} \\ \dfrac{1}{2^n} & \text{if } n \text{ is even} \end{cases}$$

It follows that for each positve integer n we have

$$\sqrt[n]{a_n} = \begin{cases} \dfrac{1}{3} & \text{if } n \text{ is odd} \\ \dfrac{1}{2} & \text{if } n \text{ is even} \end{cases}$$

Thus, for each positve integer n we have $\sqrt[n]{a_n} \leq \frac{1}{2}$. It follows from Theorem 29 Part a, with $\alpha = \frac{1}{2}$, that the given series converges.

□

Note that we could not apply Corollary 30 (Cauchy's Root Test) because the limit $\lim_{n\to\infty} \sqrt[n]{a_n}$ does not exist.

Here is an alternate solution for the same problem.

Alternate Solution. Set

$$b_k = \frac{1}{2^k} = (\frac{1}{2})^k$$

We know that the series $\sum b_k$ converges (geometric series). Also,

$$a_k \leq b_k$$

for each positive integer k. It follows that the series $\sum a_k$ also converges by the Comparison Test.

□

Now, let's see that the ratio test is inconclusive for the series in this same problem.

First, suppose that k is odd. Then

$$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{2^{k+1}}}{\frac{1}{3^k}} = \frac{3^k}{2^{k+1}} = \frac{1}{2} \cdot (\frac{3}{2})^k.$$

It follows that there is a subsequence of the sequence $\{\frac{a_{k+1}}{a_k}\}$ which diverges to infinity.

Second, suppose that k is even. Then

$$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{3^{k+1}}}{\frac{1}{2^k}} = \frac{2^k}{3^{k+1}} = \frac{1}{3} \cdot (\frac{2}{3})^k.$$

It follows that there is a subsequence of the sequence $\{\frac{a_{k+1}}{a_k}\}$ which converges to zero.

So neither of the two ratio tests from the last lecture can be applied.

□

Finally, we go over a problem from Section 7.1 of the text (Exercise 8 on Page 301).

Problem 32. Suppose that $\sum a_k$ is a series of positive terms that converges. Prove that $\sum \frac{1}{a_k}$ is divergent. Is the converse true?

Solution. Suppose that $\sum a_k$ is a series of positive terms that converges. By the Test for Divergence we have

$$\lim_{k\to\infty} a_k = 0.$$

It follows that

$$\lim_{k\to\infty} \frac{1}{a_k} = \infty.$$

So, the series $\sum \frac{1}{a_k}$ is divergent, by the Test for Divergence.

Now, let's look at the converse. I would state the converse as follows:

Converse. Suppose that $\sum a_k$ is a series of positive terms. Suppose that the series $\sum \frac{1}{a_k}$ is divergent. Then the series $\sum a_k$ converges.

We prove that this is false by giving a counterexample. Consider the series $\sum a_k$ where $a_k = 1$ for each positive integer k. Then, also $\frac{1}{k} = 1$ for each positive integer k. So both series $\sum a_k$ and $\sum \frac{1}{a_k}$ diverge, by the Test for Divergence.

□