Advanced Calculus 2, Dr. Block, Lecture Notes, 4-15-2020

We discuss material from Section 7.4 of the text. Please work on the following exercises: Section 7.4, Page 322 - 323, Exercises 2, 6, 10, 11 (all parts of each). Please email me if you have questions.

We begin with our final test for determining whether series converge or diverge.

Theorem. (Alternating Series Test) Let $\{a_n\}$ be a sequence of positive real numbers which is eventually decreasing and converges to zero. Then the alternating series $\sum(-1)^k a_k$ or $\sum(-1)^{k+1} a_k$ converges.

Please study the proof in the text on Page 318. Note that the alternating series test can only be used to show that a series converges. It can not be used be used to show that a series diverges.

The following example shows that the the conclusion of the theorem need not be true if we omit the hypothesis that the sequence $\{a_n\}$ is eventually decreasing. This example is given Exercise 11 in the text. I am calling the sequence $\{b_n\}$ instead of $\{a_n\}$ since the terms of the sequence $\{b_n\}$ alternate between positive and negative, while in the statement of the alternating series test, the terms in the sequence $\{a_n\}$ are all positive.

Problem. Suppose that $\{b_n\}$ is the sequence defined by

$$b_n = \begin{cases} \dfrac{-1}{n^2} \text{ if } n \text{ is odd} \\ \dfrac{1}{n} \text{ if } n \text{ is even} \end{cases}$$

Show that the alternating series $\sum b_k$ diverges, even though the sequence $\{b_n\}$ converges to zero.

Solution. First, consider the sequence $\{a_n\}$ given by

$$a_n = \begin{cases} \dfrac{1}{n^2} \text{ if } n \text{ is odd} \\ \dfrac{1}{n} \text{ if } n \text{ is even} \end{cases}$$

Then the series $\sum b_k$ is the same as the series $\sum(-1)^k a_k$. However, the sequence $\{a_n\}$ is not eventually decreasing. So the alternating series test can not be applied to guarantee that the series converges. It is still possible that the series converges, but it is possible that the series diverges. We will show that the series $\sum b_k$ diverges.

We can assume without loss of generality that the series $\sum b_k$ is given precisely as $\sum_{k=1}^{\infty} b_k$. For each positive integer n, let S_n denote the n-th partial sum for the series $\sum_{k=1}^{\infty} b_k$. Our goal is to show that the sequence $\{S_n\}$ diverges.

We consider two sequences $\{T_n\}$ and $\{W_n\}$ given by:

$$T_n = -\frac{1}{1^2} - \frac{1}{3^2} - \cdots - \frac{1}{(2n-1)^2}$$

$$W_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}$$

Then, for each positive integer n we have

$$S_{2n} = T_n + W_n.$$

Now, observe that T_n is the n-th partial sum of the series $\sum_{k=1}^{\infty} -\frac{1}{(2k-1)^2}$. We can see that this series converges. It follows that the sequence $\{T_n\}$ converges.

Also, observe that W_n is the n-th partial sum of the series $\sum_{k=1}^{\infty} \frac{1}{2k}$. We can see that this series diverges. It follows that the sequence $\{W_n\}$ diverges.

It follows from the lemma below that the sequence $\{S_{2n}\}$ diverges. Since the sequence $\{S_{2n}\}$ is a subsequence of the sequence $\{S_n\}$, we can conclude that the sequence $\{S_n\}$ also diverges. We conclude that the series $\sum b_k$ diverges.

$\square$

We used the following lemma about sequences in the solution above.

Lemma. Suppose that a sequence $\{x_n\}$ converges, and a sequence $\{y_n\}$ diverges. Then the sequence $\{x_n + y_n\}$ diverges.

Proof. Proceeding by contradiction suppose that the sequence

$$\{x_n + y_n\}$$

converges. Since the sequence $\{x_n\}$ also converges, it follows that the sequence $\{x_n + y_n - x_n\}$ converges. Thus, the sequence $\{y_n\}$ converges. This is a contradiction.

□

We now consider the following definition.

Definition. Let $\sum a_k$ be a series. If the series $\sum |a_k|$ converges, we say that the series $\sum a_k$ **converges absolutely.** If the series $\sum a_k$ converges and the series $\sum |a_k|$ diverges, we say that the series $\sum a_k$ **converges conditionally.**

Recall that we proved earlier, that if $\sum |a_k|$ converges, then the series $\sum a_k$ also converges. So, if a series converges abolutely, we can conclude that the series converges.

An example of a series which converges conditionally is the series $\sum(-1)^k\frac{1}{k}$. The series converges by the alternating series test. But the series $\sum |(-1)^k\frac{1}{k}|$ is the harmonic series $\sum \frac{1}{k}$, which we know diverges.

□