**Advanced Calculus 2, Dr. Block, Lecture Notes, 4-17-2020**

We will go over three problems.

**Problem 1.** Determine whether the given series converges or diverges.

$$\sum \sin(\ln(1 + \frac{1}{k^2}))$$

Solution: We try to use the Limit Comparison Test (Theorem 25 in the Lecture Notes 4-8-2020). Let $\sum a_k$ be the given series, and let $\sum b_k$ be the series $\sum \frac{1}{k^2}$. We have

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{t\to 0^+} \frac{\sin(\ln(1+t))}{t}$$

$$= \text{(L'Hopital)} \lim_{t\to 0+} (\cos(\ln(1+t)) \cdot \frac{1}{1+t}) = 1.$$

So by Theorem 25 Part a (in the Lecture Notes 4-8-2020) either both series converge or both series diverge. Since we know that $\sum b_k$ converges (as it is a $p$-series with $p = 2 > 1$; see Theorem 21, (in the Lecture Notes 4-6-2020), it follows that the given series also converges.

**Problem 2.** Determine whether the given series converges or diverges.

$$\sum_{k=2}^{\infty} \frac{\int_1^k e^{-x^2}\,dx}{k^3 - 5}$$

Solution: Let $c$ denote the value of the improper integral $\int_1^\infty e^{-x^2}\,dx$. Then $c$ is a real number, since we have seen that the improper integral converges. Moreover, since $e^{-x^2} > 0$ for all $x$, it follows that $c > 0$ and for each $k \geq 2$, we have

$$0 \leq \int_1^k e^{-x^2}\,dx \leq c.$$

Hence, we have for each integer $k \geq 2$,

$$0 \leq \frac{\int_1^k e^{-x^2}\,dx}{k^3 - 5} \leq \frac{c}{k^3 - 5}.$$

So, if we can show that the series $\sum_{k=2}^{\infty} \frac{c}{k^3-5}$ converges, it will follow that the given series also converges by the Comparison Test (Theorem 22, Part a, in the lecture Notes 4-8-2020).

We show that the series $\sum_{k=2}^{\infty} \frac{c}{k^3-5}$ does indeed converge. We use the Limit Comparison Test and compare this series with the $p$-series $\sum \frac{1}{k^3}$. We have

$$\lim_{k\to\infty} \frac{\frac{c}{k^3-5}}{\frac{1}{k^3}} = \lim_{k\to\infty} \frac{ck^3}{k^3-5} = c.$$

Since the $p$-series $\sum \frac{1}{k^3}$ converges, it follows by Theorem 25 Part a (in the Lecture Notes 4-8-2020) that the series $\sum_{k=2}^{\infty} \frac{c}{k^3-5}$ also converges.

We conclude by what we saw above that the series

$$\sum_{k=2}^{\infty} \frac{\int_1^k e^{-x^2}\, dx}{k^3-5}$$

converges.

**Problem 3.** Determine whether the given series converges absolutely, converges conditionally, or diverges.

$$\sum (-1)^k \sin(\frac{1}{k})$$

Solution: First, we test for absolute convergence. We consider the series

$$\sum |(-1)^k \sin(\frac{1}{k})|.$$

Since, $\sin \frac{1}{k} > 0$ for all $k \geq 1$ we see that this series is the same as the series $\sum \sin \frac{1}{k}$.

We use the Limit Comparison Test, comparing this series with the series $\sum \frac{1}{k}$. We have

$$\lim_{k\to\infty} \frac{\sin(\frac{1}{k})}{\frac{1}{k}} = \lim_{t\to 0^+} \frac{\sin t}{t} = 1.$$

We know that the harmonic series $\sum \frac{1}{k}$ diverges (Theorem 11 in the Lecture Notes 4-1-2020). (Also, the harmonic series is a $p$-series with $p = 1$, so we could apply Theorem 21 in the Lecture Notes 4-6-2020.) It follows from Theorem 25 Part a (in the Lecture Notes 4-8-2020) that the series $\sum \sin \frac{1}{k}$ also diverges. We conclude that the given series does not converge absolutely.

Since $\sin(\frac{1}{k}) > 0$ for all $k \geq 1$, the given series is an alternating series. We check that the hypotheses of the Alternating Series Test (in the Lecture Notes 4-15-2020) are satisfied. First, we observe that $\lim_{k\to\infty} \sin(\frac{1}{k}) = 0$.

Next we show that the sequence $\{\sin(\frac{1}{k})\}$ is decreasing. We consider the function $f(x) = \sin(\frac{1}{x})$. We see that $f'(x) = \frac{-\cos(\frac{1}{x})}{x^2} < 0$ for all $x \geq 1$. It follows that the sequence $\{\sin \frac{1}{k}\}$ is decreasing. So the given series converges by the Alternating Series Test.

Finally, since the given series converges, but does not converge absolutely, it follows by definition that the given series converges conditionally. $\square$