Advanced Calculus 2, Dr. Block, Lecture Notes, 4-20-2020

We will go over four additional problems. We give two solutions for the first problem. Many problems on convergence of series could be solved by several different methods.

1. Determine whether the given series converges or diverges.

$$\sum(-1)^k \frac{k!}{k^k}$$

First Solution: We check for absolute convergence. Let $\sum a_k$ be the series given by

$$a_k = |(-1)^k \frac{k!}{k^k}| = \frac{k!}{k^k}.$$

We use the ratio test. We have

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)! \cdot k^k}{k! \cdot (k+1)^{(k+1)}} = \frac{(k+1) \cdot k^k}{(k+1)^{(k+1)}}$$

$$= \frac{k^k}{(k+1)^k} = (\frac{k}{k+1})^k = \frac{1}{(\frac{k+1}{k})^k} = \frac{1}{(1+\frac{1}{k})^k}.$$

It follows that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{e}.$$

Since $\frac{1}{e} < 1$, the series $\sum a_k$ converges by Cauchy's Ratio Test (Corollary 26, Part a, Lecture notes 4-10-2020). It follows that the given series converges absolutely, and hence converges.

Second Solution: We check for absolute convergence. Let $\sum a_k$ be the series given by

$$a_k = |(-1)^k \frac{k!}{k^k}| = \frac{k!}{k^k}.$$

We have

$$a_k = \frac{k(k-1)(k-2)\dots(2)(1)}{k \cdot k \cdot k \cdot \dots \cdot k} = (\frac{k}{k})(\frac{k-1}{k})(\frac{k-2}{k})\cdots(\frac{2}{k})(\frac{1}{k}).$$

It follows that $a_k \le \frac{2}{k^2}$ for all $k \ge 2$. We know that the series $\sum \frac{1}{k^2}$ converges, as it is a p-series with $p = 2 > 1$. Hence the series $\sum \frac{2}{k^2}$ also converges. So the series $\sum a_k$ converges by the Comparison Test. It follows that the given series converges absolutely, and hence converges.

2. Determine whether the given series converges or diverges.

$$\sum (-1)^k \frac{k^k}{k!}$$

Solution: We observe that $\frac{k^k}{k!} \geq 1$ for each positive integer k. So, the sequence $\{\frac{k^k}{k!}\}$ does not converge to zero. It follows that the sequence $\{(-1)^k \frac{k^k}{k!}\}$ does not converge to zero. Thus, the given series diverges, by the Test for Divergence.

$\square$

3. Determine whether the given series or diverges.

$$\sum a_k$$

where

$$a_k = \begin{cases} \dfrac{k^3}{2^k} & \text{if } k \text{ is even} \\ \dfrac{1}{3^k} & \text{if } k \text{ is odd} \end{cases}$$

Solution: We claim that the $\sqrt[k]{a_k} \leq \frac{3}{4}$ for all k sufficiently large. If k is odd then $\sqrt[k]{a_k} = \frac{1}{3} < \frac{3}{4}$ Now we consider the case where k is even. We observe that

$$\lim_{k \to \infty} \sqrt[k]{\frac{k^3}{2^k}} = \lim_{k \to \infty} \frac{1}{2} \cdot (\sqrt[k]{k})^3 = \frac{1}{2}.$$

Hence, $\sqrt[k]{\frac{k^3}{2^k}} < \frac{3}{4}$ for all k sufficiently large. This establishes the claim.

So we may apply the Root Test (Theorem 29, Part a, in the Lecture notes 4-13-2020), and conlude that the given series converges.

$\square$

4. Determine whether the given series or diverges.

$$\sum_{k=2}^{\infty} \frac{1}{(\ln k)^2}$$

Solution: We try the Limit Comparison Test, and compare the given series with the harmonic series $\sum \frac{1}{k}$. We have

$$\lim_{k\to\infty} \frac{\frac{1}{(\ln k)^2}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{(\ln k)^2} = (L'Hopital) \lim_{k\to\infty} \frac{1}{2(\ln k)(\frac{1}{k})}$$
$$= \lim_{k\to\infty} \frac{k}{2\ln k} = (L'Hopital) \lim_{k\to\infty} \frac{1}{2(\frac{1}{k})} = \lim_{k\to\infty} \frac{k}{2} = \infty.$$

Since the harmonics eries $\sum \frac{1}{k}$ diverges, the given series also diverges, by the Limit Comparison Test (Theorem 25, Part C, in the Lecture Notes 4-8-2020).

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