Advanced Calculus 2, Dr. Block, Lecture Notes, 4-22-2020

We look at the following example. We will prove that a certain sequence converges. It turns out that the limit of this sequence is a special real number, called the Euler, Mascheroni constant (also called Euler's constant). There are several different infinite series which converge to this constant. You can find some of these and other information about this constant on Wikipedia.

Problem. For each positive integer k, let

$$a_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \ln k.$$

Prove that the sequence $\{a_k\}$ converges.

Solution. Our strategy is to prove that the sequence is decreasing and each a_k is positive. It follows that the sequence is monotone and bounded, and hence the sequence converges. We will do this by proving three claims.

Claim 1. For each positive integer k, we have

$$\frac{1}{k} \geq \int_k^{k+1} \frac{1}{x} dx \geq \frac{1}{k+1}.$$

Proof of Claim 1. Consider the functions f, g and h defined on the interval $[k, k+1]$ by

$$h(x) = \frac{1}{k},\ f(x) = \frac{1}{x},\ \text{and}\ g(x) = \frac{1}{k+1}.$$

Observe that for all $x \in [k, k+1]$, we have

$$h(x) \geq f(x) \geq g(x).$$

So, by one of our theorems on integration, it follows that

$$\int_k^{k+1} h(x)dx \geq \int_k^{k+1} f(x)dx \geq \int_k^{k+1} g(x)dx.$$

Therefore,

$$\frac{1}{k} \geq \int_k^{k+1} \frac{1}{x} dx \geq \frac{1}{k+1}.$$

This proves the Claim 1.

Claim 2. For each positive integer k, we have $a_{k+1} - a_k \leq 0$.

Proof of Claim 2. Suppose that k is a positive integer. Then

$$a_{k+1} - a_k = \frac{1}{k+1} - \ln(k+1) + \ln k = \frac{1}{k+1} - \int_k^{k+1} \frac{1}{x} dx.$$

It follows from this equality and Claim 1, that $a_{k+1} - a_k \leq 0$. This proves Claim 2.

It follows from Claim 2 that the sequence $\{a_k\}$ is decreasing.

Claim 3. For each psoitive integer k, we have

$$a_k \geq \frac{1}{k} > 0.$$

Proof of Claim 3. Suppose that k is a positive integer. Then

$$a_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \ln k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \int_1^k \frac{1}{x} dx$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - (\int_1^2 \frac{1}{x} dx + \int_2^3 \frac{1}{x} dx + \cdots + \int_{k-1}^k \frac{1}{x} dx)$$

$$= (1 - \int_1^2 \frac{1}{x} dx) + (\frac{1}{2} - \int_2^3 \frac{1}{x} dx) + \cdots + (\frac{1}{k-1} - \int_{k-1}^k \frac{1}{x} dx) + \frac{1}{k}.$$

It follows from Claim 1 that each of the terms in parentheses in the above sum is greater than or equal to zero. Thus,

$$a_k \geq \frac{1}{k} > 0.$$

This proves Claim 3. It follows that the sequence $\{a_k\}$ is bounded below by zero. Since the sequence is decreasing, we can conclude that the sequence converges.

□