Advanced Calculus 2, Dr. Block, Lecture Notes, 4-3-2020

We continue discussing material from Section 7.1 of the text. Please work on this assignment:

Section 7.1, Page 300 - 302, Exercises 1, 4, 5, 8, 9, 10, 16, 19, 20 (all parts of each).

First we have a remark.

Remark 14. Suppose that we are given a sequence $\{a_n\}_{n=1}^{\infty}$ and consider the corresponding series $\sum_{n=1}^{\infty} a_n$. Let's also look at the series $\sum_{n=3}^{\infty} a_n$ and compare this to the series $\sum_{n=1}^{\infty} a_n$.

Let V_j denote the j-th partial sum for the series $\sum_{n=1}^{\infty} a_n$.

Let W_j denote the j-th partial sum for the series $\sum_{n=3}^{\infty} a_n$. Then we have for any positive integer j:

$$W_j = a_3 + a_4 + a_5 + \cdots + a_{j+2}.$$

$$V_{j+2} = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_{j+2} = (a_1 + a_2) + W_j.$$

It follows that the sequence $\{V_j\}$ converges if and only if the sequence $\{W_j\}$ converges. So, by definition, the series $\sum_{n=1}^{\infty} a_n$ coverges if and only if the series $\sum_{n=3}^{\infty} a_n$ converges. Moreover, we see that if both series converge, then we can write

$$\sum_{n=1}^{\infty} a_n = (a_1 + a_2) + \sum_{n=3}^{\infty} a_n..$$

In the same way we can see that for any integer $p \geq 2$, the series $\sum_{n=1}^{\infty} a_n$ coverges if and only if the series $\sum_{n=p}^{\infty} a_n$ converges. For this reason, when dealing with theorems or problems involving determining whether a series converges or diverges, the place where we start the series is not important. So, we will often just write $\sum a_n$ instead of $\sum_{n=p}^{\infty} a_n$.

We next look at two theorems which we will use later.

Theorem 15. Let $\sum a_n$ be a series such that $a_n \geq 0$ for each n. Then the series $\sum a_n$ converges if and only if the sequence of partial sums is bounded.

Proof. The hypothesis implies that the sequence of partial sums in increasing. So the conclusion follows from two basic theorems about sequences:

1. A bounded increasing sequence converges.
2. A convergent sequence is bounded.

□

For the next theorem we will use a result from last time. Recall the following:

Theorem 12. Suppose that $\sum_{n=p}^{\infty} a_n = A$ and $\sum_{n=p}^{\infty} b_n = B$, where A and B are real numbers. Then

$$\sum_{n=p}^{\infty}(a_n + b_n) = A + B$$

and

$$\sum_{n=p}^{\infty}(a_n - b_n) = A - B.$$

Also, if c is a real constant then,

$$\sum_{n=p}^{\infty} ca_n = cA.$$

We can now prove the following theorem.

Theorem 16. If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Suppose that the series $\sum_{n=1}^{\infty} |a_n|$ converges. Then by Theorem 15, there is an upper bound B for the sequence of partial sums of this series.

Consider the sequence $\{b_n\}$ given by $b_n = |a_n| - a_n$. We have for each $n \geq 1$,

$$0 \leq b_n \leq 2|a_n|.$$

So, for any positive integer k we have

$$0 \leq b_1 + b_2 + \cdots + b_k \leq 2(|a_1| + +|a_2| + \cdots + |a_k|) \leq 2B.$$

It follows that the sequence of partial sums for the series $\sum_{n=1}^{\infty} b_n$ is bounded. Hence, the series $\sum_{n=1}^{\infty} b_n$ converges.

Now, as $b_n = |a_n| - a_n$, we have $a_n = |a_n| - b_n$. So, by Theorem 12 the series $\sum_{n=1}^{\infty} a_n$ converges.

$\square$