Advanced Calculus 2, Dr. Block, Lecture Notes, 4-6-2020

We begin discussing material from Section 7.2 of the text. Please work on this assignment:

Section 7.2, Page 309 - 311, Exercises 1, 5, 8, 10, 15, 16 (all parts of each).

In this section we have three useful tests for determining whether series converge or diverge. We discuss the first of the three tests today.

Theorem 17 (Integral Test). Let $\sum_{k=1}^{\infty} a_k$ be a series. Suppose that $f : [1, \infty) \to \mathbb{R}$ is continuous, nonnegative, and decreasing and also $f(k) = a_k$ for each postive integer k. Then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges.

Please study the proof of this theorem given in the text on pages 303 and 304. Note that in the statement of the theorem the series started with k=1. But we can easily see that the theorem is valid if the series starts at any nonnegative integer. To use this theorem we need to find a function f which satisfies all of the hypotheses of the theorem. This is only possible when each $a_k \geq 0$. Also, we can only apply the theorem in cases where we can determine whether the improper integral converges or diverges. Here is an example where the integral test may be used.

Problem 18. Determine whether the given series converges or diverges.

$$\sum_{k=2}^{\infty} \frac{1}{k \ln k}$$

Solution. We apply the integral test. Consider the function $f : [2, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x \ln x}$. It is evident that $f(k) = a_k$ for each integer $k \geq 2$. Also, f is continuous, and for all $x \geq 2$ we have $f(x) > 0$.

We prove that f is decreasing. Suppose that s and t are real numbers with $2 \leq s < t$. Since the function $\ln x$ is strictly increasing, we have

$\ln s < \ln t$. It follows that $s \ln s < t \ln t$. Finally, since the function $\frac{1}{x}$ is strictly decreasing, we have

$$f(s) = \frac{1}{s \ln s} > \frac{1}{t \ln t} = f(t).$$

So, f is decreasing.

Since all of the hypotheses of the integral test are satisfied, we can now conclude that the series $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ converges if and only if the improper integral $\int_1^{\infty} \frac{1}{x \ln x} dx$ converges.

It remains to evaluate the improper integral. We have

$$\int_1^{\infty} \frac{1}{x \ln x} dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x \ln x} dx$$

$$= \lim_{b \to \infty} (\ln(\ln x)\Big|_2^b) = \lim_{b \to \infty} [(\ln(\ln b) - \ln(\ln 2)] = \infty.$$

Since the improper integral diverges, it follows from the integral test that the series also diverges.

$\square$

Remark 19. In the previous example, we proved that the function f was decreasing. An alternate way we could have done this is to prove that the derivative of f is negative. Also, the integral test is valid if the function f is eventually decreasing (eventually decreasing means decreasing on some interval $[w, \infty)$). So it is sufficient to prove that the derivative of f is eventually negative.

Definition 20. Let p be a positive real number. The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p-series.

We have the following theorem.

Theorem 21. A p-series converges if $p > 1$ and diverges if $p \leq 1$.

Proof. Consider the function $f : [1, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x^p}$. Then f is continuous, nonnegative, and decreasing, and also $f(k) = \frac{1}{k^p}$

for each postive integer k. It follows that the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if the improper integral $\int_1^{\infty} \frac{1}{x^p} dx$ converges. We know from Section 6.5 that this improper integral converges if $p > 1$ and diverges if $p \leq 1$. It follows that the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

$\square$