Advanced Calculus 2, Dr. Block, Lecture Notes, 4-8-2020

We continue discussing material from Section 7.2 of the text. Please work on this assignment:

Section 7.2, Page 309 - 311, Exercises 1, 5, 8, 10, 15, 16 (all parts of each).

Theorem 22. (Comparison Test) Let $\sum a_k$ and $\sum b_k$ be series. Suppose that there is a positive integer J such that $0 \leq a_n \leq b_n$ for all $n \geq J$.

a. If $\sum b_k$ converges, then $\sum a_k$ converges.

b. If $\sum a_k$ diverges, then $\sum b_k$ diverges.

Proof. We prove part a. We may assume, without loss of generality (by Remark 13 in the Lecture notes for 4-1-2020), that both series begin with the integer J.

Suppose that $\sum_{k=J}^{\infty} b_k$ converges.
Let W_n denote the n-th partial sum for the series $\sum_{k=J}^{\infty} b_k$.
Let S_n denote the n-th partial sum for the series $\sum_{k=J}^{\infty} a_k$.

Since $0 \leq a_n \leq b_n$ for all $n \geq J$, it follows that for each positive integer n, we have

$$0 \leq S_n \leq W_n.$$

By Theorem 15 (in the Lecture notes for 4-3-2020), the sequence $\{W_n\}$ is bounded. By the displayed inequality the sequence $\{S_n\}$ is also bounded. It follows from Theorem 15 that the series $\sum a_k$ converges. This proves part a.

Finally, observe that the statement in Part b is the contrapositive of the statement in Part a. So, Part b follows from Part a.

□

Remark 23. We know that the sequence $\{(1 + \frac{1}{k})^k\}$ converges to e. It can be proved that this sequence is increasing. It follows that for each positive integer k, we have

$$(1+\frac{1}{k})^k \leq e.$$

We will use this fact in our next example.

Problem 24. Determine whether the given series converges or diverges.

$$\sum \frac{1}{(1+\frac{1}{k})^{k\ln k}}$$

Solution. We start with the displayed inequality in Remark 23 above. Taking each side of the inequality and raising to the power $\ln k$ we have

$$(1+\frac{1}{k})^{k\ln k} \leq e^{\ln k} = k.$$

It follows that

$$\frac{1}{(1+\frac{1}{k})^{k\ln k}} \geq \frac{1}{k}.$$

We know that the series $\sum \frac{1}{k}$ diverges, as this is a p-series with $p=1$. It follows that the given series diverges by the Comparison Test.

□

We now come to our second test for today.

Theorem 25 (Limit Comparison Test) Let $\sum a_n$ and $\sum b_n$ be series with all terms positive.

a. Suppose that there is a positive real number c such that $\lim_{n\to\infty} \frac{a_n}{b_n} = c$. Then either both series converge or both series diverge.

b. Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. If $\sum b_n$ converges, then $\sum a_n$ converges.

c. Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$. If $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof of Part a. Suppose that there is a positive real number c such that $\lim_{n\to\infty} \frac{a_n}{b_n} = c$. Then there is a positive integer J such that for each integer $n \geq J$ we have:

$$0 \leq \frac{c}{2} \leq \frac{a_n}{b_n} \leq 2c$$

It follows that for each integer $n \geq J$ we have $\frac{c}{2} \cdot b_n \leq a_n \leq 2c \cdot b_n$. We consider 2 cases.

Case 1. The series $\sum b_n$ converges. Then the series $\sum 2c \cdot b_n$ also converges. Hence the series $\sum a_n$ converges by the Comparison Test.

Case 2. The series $\sum b_n$ diverges. Then the series $\sum \frac{c}{2} \cdot b_n$ also diverges. Hence the series $\sum a_n$ diverges by the Comparison Test.

We conclude from the two cases that either both series converge or both series diverge.

Proof of Part b. Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and also $\sum b_n$ converges. Then there is a positive integer J such that for each integer $n \geq J$ we have $\frac{a_n}{b_n} \leq 1$. It follows that for each integer $n \geq J$ we have $a_n \leq b_n$. So the series $\sum a_n$ converges by the Comparison Test.

Proof of Part c. Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and also $\sum b_n$ diverges. Then there is a positive integer J such that for each integer $n \geq J$ we have $\frac{a_n}{b_n} \geq 1$. It follows that for each integer $n \geq J$ we have $a_n \geq b_n$. So the series $\sum a_n$ diverges by the Comparison Test.

□

Here is an example.

Problem 24. Determine whether the given series converges or diverges.

$$\sum \sin(\frac{1}{k})$$

Solution. We use the Limit Comparison Test, comparing the given series with the series $\sum \frac{1}{k}$. We have

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{t\to 0} \frac{\sin t}{t} = 1.$$

By the Limit Comparison Test, Part a, either both series converge or both series diverge. Since we know that the series $\sum \frac{1}{k}$ diverges, we conclude that the series $\sum \sin(\frac{1}{k})$ also diverges.

$\square$