Advnced Calculus 2, Dr. Block, Lecture Notes, 3-18-2020

We begin Section 6.5. First, we look at Definition 27 from the Chapter 6 notes.

Definition 27. Let $a \in \mathbb{R}$, and let $f : [a, \infty) \to \mathbb{R}$. Suppose that f is Riemann integrable on $[a, b]$ for each $b > a$. If $\lim_{b\to\infty} \int_a^b f$ exists and is some real number L, then we say that the improper integral $\int_a^\infty f$ converges to L and write $\int_a^\infty f = L$.

Here is an Example.

Problem 1. For which $p > 0$ does the integral $\int_1^\infty \frac{1}{x^p}\, dx$ converge?

Solution. We consider three cases.

Case 1. $p > 1$. Then

$$\int_1^\infty \frac{1}{x^p}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\, dx = \lim_{b\to\infty} (\frac{x^{1-p}}{1-p}\Big|_1^b)$$

$$= \lim_{b\to\infty} (\frac{b^{1-p}}{1-p} - \frac{1}{1-p}) = \frac{1}{p-1}.$$

So in this case the improper integral converges.

Case 2. $p < 1$. Then

$$\int_1^\infty \frac{1}{x^p}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\, dx = \lim_{b\to\infty} (\frac{x^{1-p}}{1-p}\Big|_1^b)$$

$$= \lim_{b\to\infty} (\frac{b^{1-p}}{1-p} - \frac{1}{1-p}) = \infty.$$

So in this case the improper integral diverges.

Case 3. $p = 1$. Then

$$\int_1^\infty \frac{1}{x^p}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\, dx = \lim_{b\to\infty} \ln x\Big|_1^b$$

$$= \lim_{b\to\infty} \ln b = \infty.$$

So in this case the improper integral diverges.

We conclude that the integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \leq 1$.

□

The conclusion of the previous problem will be useful to remember when using the Comparison Test (Theorem 30 in the Chapter 6 notes), which we will discuss later. We now look at another type of improper integral.

Definition 28. Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b] \to \mathbb{R}$ be a function which is not bounded. Suppose that f is Riemann integrable on $[c, b]$ for each c in the open interval (a, b). If $\lim_{c\to a^+} \int_c^b f$ exists and is some real number L, then we say that the improper integral $\int_a^b f$ converges to L and write $\int_a^b f = L$.

Note that in this definition we take the limit as c approaches a from the right of a. So c approaches a from inside the interval $(a, b]$.

Here is an Example.

Problem 2. For which $p > 0$ does the integral $\int_0^1 \frac{1}{x^p}\,dx$ converge?

Solution. We consider three cases.

Case 1. $p > 1$. Then

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+} \left(\frac{x^{1-p}}{1-p}\bigg|_c^1\right)$$

$$= \lim_{c\to 0^+} \left(\frac{1}{1-p} - \frac{c^{1-p}}{1-p}\right) = \infty.$$

So in this case the improper integral diverges.

Case 2. $p < 1$. Then

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+} \left(\frac{x^{1-p}}{1-p}\bigg|_c^1\right)$$

$$= \lim_{c\to 0^+} \left(\frac{1}{1-p} - \frac{c^{1-p}}{1-p}\right) = \frac{1}{1-p}.$$

So in this case the improper integral converges.

Case 3. $p = 1$. Then

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \ln x \Big|_c^1$$

$$= \lim_{c\to 0^+} -\ln c = -(-\infty) = \infty.$$

So in this case the improper integral converges.

We conclude that the integral $\int_0^1 \frac{1}{x^p}\,dx$ converges if $p < 1$ and diverges if $p \geq 1$.

□

Again, the conclusion of the previous problem will be useful to remember when using the Comparison Test (Theorem 30 in the Chapter 6 notes), which we will discuss later.

We sometimes consider improper integrals which are improper in two ways. Here is an example.

Problem 3. For which $p > 0$ does the integral $\int_0^\infty \frac{1}{x^p}\,dx$ converge?

Note that the given function is unbounded at 0. Also, note that the upper limit of integration is ∞. We have not yet defined what it means for this improper integral to converge. The definition amounts to the following:

The improper integral $\int_0^\infty \frac{1}{x^p}\,dx$ converges if and only if **both of the improper integrals** $\int_0^1 \frac{1}{x^p}\,dx$ and $\int_1^\infty \frac{1}{x^p}\,dx$ converge. So we split the given improper integral into two improper integrals each of which is improper in one way only. Instead of splitting the integrals using the real number 1 we could use $2, 3$ or any positive real number.

So the answer to Problem 3 follows from the answers to Problems 1 and 2. The improper integral $\int_0^\infty \frac{1}{x^p}\,dx$ diverges for all $p > 0$.

□