Advanced Calculus 2, Dr. Block

Lecture Notes, 3-25-2020

We go over three examples.

Problem 10 d section 6.5, $\int_0^1 \frac{dx}{x(x+1)}$

Note that this is a notation for

$$\int_0^1 \frac{1}{x(x+1)}\,dx.$$

Solution: Note that for $x \in [0, 1]$,

$1 \leq x+1 \leq 2$. So $\frac{1}{2} \leq \frac{1}{x+1} \leq 1$.

So, we have a lower bound and an upper bound for $\frac{1}{x+1}$. Here, we use the lower bound.

Since $\frac{1}{x+1} \geq \frac{1}{2}$, we have $\frac{1}{x(x+1)} \geq \frac{1}{2} \cdot \frac{1}{x}$.

We know that $\int_0^1 \frac{1}{x}\,dx$ diverges.

So $\int_0^1 \frac{1}{2} \cdot \frac{1}{x}\,dx$ also diverges.

Thus, $\int_0^1 \frac{1}{x(x+1)}\,dx$ also diverges.

$\square$

Problem 14 Section 6.5

Evaluate $\lim_{x \to \infty} \frac{1}{x} \int_1^x \sqrt{4 + \exp(-t)}\, dt$

if the limit exists.

Solution: Note that as $\exp(-t) \geq 0$ for all t, we have

$$\sqrt{4 + \exp(-t)} \geq 2 \text{ for all } t.$$

So by comparison, the improper integral $\int_1^\infty \sqrt{4 + \exp(-t)}\, dt$ diverges.

Thus, $\lim_{x \to \infty} \int_1^x \sqrt{4 + \exp(-t)}\, dt = \infty.$

We may write the given limit as

$$\lim_{x \to \infty} \frac{\int_1^x \sqrt{4 + \exp(-t)}\, dt}{x}$$

This is of the form $\frac{\infty}{\infty}$. By L'Hopital's rule the limit is

$$\lim_{x \to \infty} \frac{\sqrt{4 + \exp(-x)}}{1} = 2.$$

$\square$

Here is another example.

Evaluate $\displaystyle \lim_{x \to \infty} \frac{1}{x} \int_1^x \frac{t + \sqrt{t} + 5}{t^3}\, dt.$

Solution: We see that

$$\frac{t+\sqrt{t}+5}{t^3} \le \frac{t+t+5t}{t^3} = \frac{7t}{t^3} = 7\left(\frac{1}{t^2}\right)$$

for all $t \ge 1$. So the improper integral $\int_1^\infty \frac{t+\sqrt{t}+5}{t^3}\,dt$ converges by comparison.

Thus, $\lim\limits_{x\to\infty} \int_1^x \frac{t+\sqrt{t}+5}{t^3}\,dt = k$

for some real number k.

It follows that

$$\lim_{x\to\infty} \frac{1}{x}\int_1^x \frac{t+\sqrt{t}+5}{t^3}\,dt = 0\cdot k = 0.$$

$\square$