Introduction to Complex Variables
Class Notes
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Definition 1. (and remark) We consider the complex plane consisting of all \( z = (x, y) = x + iy \), where \( x \) and \( y \) are real. We write \( x = Re z \) (the real part of \( z \)) and \( y = Im z \) (the imaginary part of \( z \)). We define addition and multiplication so that the axioms of a field are satisfied. The complex number \( i \) satisfies \( i^2 = -1 \). Note that any real number \( x \) is also a complex number \( x = x + i0 \). Note also that for any complex number \( z \), we have \( 0 \cdot z = 0 \).

Definition 2. Let \( z = x + iy \) where \( x, y \) are real. The complex conjugate of \( z \) is given by \( \bar{z} = x - iy \). The absolute value of \( z \) is given by \( |z| = \sqrt{x^2 + y^2} \).

Proposition 3. Let \( z \) and \( w \) be complex numbers.
1. \( z + \bar{w} = \bar{z} + w \).
2. \( z - w = \bar{z} - \bar{w} \).
3. \( z \cdot w = \bar{z} \cdot \bar{w} \).
4. \( \overline{\frac{z}{w}} = \frac{\bar{z}}{\bar{w}} \).
5. \( z \cdot \bar{z} = |z|^2 \).
6. \( |z + w| \leq |z| + |w| \).
7. \( |z \cdot w| = |z| \cdot |w| \).
8. \( |\frac{z}{w}| = \frac{|z|}{|w|} \).

Remark 4. Let \( z \) and \( w \) be complex numbers. Then \( |z - w| \) is the usual distance between \( z \) and \( w \) considered as points in the plane.

Definition 5. (and remark) For any real number \( \theta \) we set \( e^{i\theta} = \exp(i\theta) = \cos \theta + i \sin \theta \). Then any complex number \( z \) can be written as \( z = re^{i\theta} \), where \( r = |z| \). Any such \( \theta \) is called an argument of \( z \). The set arguments of \( z \) is denoted by \( \arg(z) \). The principal argument of \( z \), denoted by \( \text{Arg}(z) \) is the unique \( \theta \) in \( \arg(z) \) with \(-\pi < \theta \leq \pi\).

Proposition 6. Let \( z_1 = r_1 e^{i\theta_1} \) \( z_2 = r_2 e^{i\theta_2} \) be complex numbers with \( r_1 > 0, r_2 > 0 \).
1. \( z_1 = z_2 \) if and only if \( r_1 = r_2 \) and \( \theta_1 = \theta_2 + 2k\pi \) for some integer \( k \).
2. \( z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \).
3. \( \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \).

Proposition 7. If \( z = re^{i\theta} \) and \( k \) is a positive integer, then \( z^k = r^k e^{ik\theta} \).

Proposition 8. Let \( z_1 \) and \( z_2 \) be complex numbers. Then \( \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \).

Proposition 9. (and Definition) Let \( z_0 \) be a non-zero complex number with \( z_0 = r_0 e^{i\theta_0} \) and \( r_0 = |z_0| \). Let \( n \) be a positive integer. A complex number \( z \) satisfies \( z^n = z_0 \) if and only if for some \( k = 0, 1, \ldots, n - 1 \) we have \( z = \sqrt[n]{r_0} \exp(i(\frac{\theta_0}{n} + \frac{2k\pi}{n})) \).

We call these complex numbers \( z \), the \( n \)-th roots of \( z_0 \). In the special case \( z_0 = 1 \), we call these complex numbers \( z \), the \( n \)-th roots of unity. Note that the symbol \( \sqrt[n]{r_0} \) denotes the unique positive real number which is an \( n \)-th roots of \( r_0 \). We let \( \left( \frac{z}{z_0} \right)^{\frac{1}{n}} \) denote the set of \( n \)-th roots of \( z_0 \).

Definition 10. (and remark) We will sometimes refer to complex numbers as points. Let \( z_0 \) be a point.
The \( \epsilon \) neighborhood of \( z_0 \) is the set of points given by \( |z - z_0| < \epsilon \).
The deleted \( \epsilon \) neighborhood of \( z_0 \) is the set of points given by \( 0 < |z - z_0| < \epsilon \).
Definition 11. Let $S$ be a subset of the set of complex numbers. Let $z_0$ be a point.

We say that $z_0$ is an interior point of $S$ if and only if there exists an $\epsilon$ neighborhood of $z_0$ which is a subset of $S$. We say that $z_0$ is an exterior point of $S$ if and only if there exists an $\epsilon$ neighborhood of $z_0$ which is a subset of the complement of $S$.

We say that $z_0$ is a boundary point of $S$ if and only if $z_0$ is neither an interior point of $S$ nor an exterior point of $S$.

The set of boundary points of $S$ is called the boundary of $S$. We say that $S$ is open if and only if no boundary point of $S$ is an element of $S$. We say that $S$ is closed if and only if each boundary point of $S$ is an element of $S$. The closure of $S$ is the union of $S$ and the set of boundary points of $S$.

Proposition 12. Let $S$ be a subset of the set of complex numbers. $S$ is open if and only if each point of $S$ is an interior point of $S$. $S$ is closed if and only if the closure of $S$ is $S$.

Definition 13. (and remark) Let $S$ be an open subset of the set of complex numbers. We say that $S$ is connected if and only if any two points of $S$ can be joined by a finite union of line segments joined end to end that lie entirely in $S$. This is not the usual topological definition, but is equivalent in this setting.

A nonempty, connected, open set is called a domain.

Definition 14. Let $S$ be a subset of the set of complex numbers. We say that $S$ is bounded if and only if there is a positive real number $B$ such that for all $z \in S$, we have $|z| < B$.

Definition 15. Let $S$ be a subset of the set of complex numbers, and let $w$ be a complex number. We say that $w$ is an accumulation point of $S$ if and only if every deleted neighborhood of $w$ contains at least one point of $S$.

Proposition 16. Let $S$ be a subset of the set of complex numbers. $S$ is closed if and only if every accumulation point of $S$ is an element of $S$.

Definition 17. (and remark) We will let $\mathbb{C}$ denote the set of complex numbers. We will study functions $f : D \to \mathbb{C}$ where $D$ is a subset of $\mathbb{C}$. The set $D$ is called the domain of the function. We may think of a function as a rule which assigns to each element of the domain a unique complex number. If the domain is not specified, we assume the domain is the set of all complex numbers for which the rule makes sense. Observe that for any $f : D \to \mathbb{C}$, there exists a unique pair of real valued functions $u, v$ of the two variables $x, y$ such that

$$f(x + iy) = u(x, y) + iv(x, y)$$

for all complex numbers in $D$.

Definition 18. Let $f : D \to \mathbb{C}$ where $D$ is a subset of $\mathbb{C}$. Suppose that $z_0$ is a complex number, and $z_0$ is an accumulation point of $D$. Recall that this holds if there is a deleted neighborhood of $z_0$ which is a subset of $D$. Let $w_0$ be a complex number. We say that $\lim_{z \to z_0} f(z) = w_0$ if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $z \in D$ which are in the deleted $\delta$ neighborhood of $z_0$, we have that $|f(z) - w_0| < \epsilon$.

Proposition 19. Let $f : D \to \mathbb{C}$ where $D$ is a subset of $\mathbb{C}$. Suppose that $z_0 = x_0 + iy_0$ is a complex number, and $z_0$ is an accumulation point of $D$. Let $w_0 = u_0 + iv_0$ be a complex number. Then $\lim_{z \to z_0} f(z) = w_0$ if and only if both $\lim_{u(x, y) \to (x_0, y_0)} u(x, y) = u_0$ and $\lim_{v(x, y) \to (x_0, y_0)} v(x, y) = v_0$.

Remark 20. We have the same theorems for limits of sums, products, and quotients that hold for real valued functions of a real variable. See Theorem 2 page 48 of the text for a precise statement.

Definition 21. Let $f : D \to \mathbb{C}$ where $D$ is a subset of $\mathbb{C}$. Suppose that $z_0$ is a complex number and $z_0$ is an accumulation point of $D$. We say that $\lim_{z \to z_0} f(z) = \infty$ if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $z \in D$ which are in the deleted $\delta$ neighborhood of $z_0$, we have that $|f(z)| > \delta$.

Definition 22. Let $f : D \to \mathbb{C}$ where $D$ is a subset of $\mathbb{C}$. Suppose that there is a positive number $B$ such that the set of $z$ with $|z| > B$ is a subset of $D$. Let $w_0$ be a complex number. We say that $\lim_{z \to \infty} f(z) = w_0$ if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $z$ with $|z| > \frac{1}{\delta}$ we have that $|f(z) - w_0| < \epsilon$. 


Definition 23. Let $f : D \to \mathbb{C}$ where $D$ is a subset of $\mathbb{C}$. Suppose that there is a positive number $B$ such that the set of $z$ with $|z| > B$ is a subset of $D$. We say that $\lim_{z \to \infty} f(z) = \infty$ if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $z$ with $|z| > \frac{1}{\delta}$ we have $|f(z)| > \frac{1}{\epsilon}$.

Proposition 24. If $\lim_{z \to z_0} \frac{1}{f(z)} = 0$, then $\lim_{z \to z_0} f(z) = \infty$.

Proposition 28. Let $f : D \to \mathbb{C}$ where $D$ is a subset of $\mathbb{C}$. Suppose that $z_0$ is an interior point of $D$. Then $f$ is differentiable at $z_0$ if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $z \in D$ with $|z - z_0| < \delta$ we have $|f(z) - f(z_0)| < \epsilon$.

Proposition 29. Let $f : D \to \mathbb{C}$ where $D$ is a subset of $\mathbb{C}$. Suppose that $z_0 = x_0 + iy_0$ is an interior point of $D$. Suppose that $f(x + iy) = u(x,y) + iv(x,y)$ for all points of $D$. If $f$ is differentiable at $z_0$, then the first partial derivatives of $u$ and $v$ exist at $(x_0, y_0)$ and we have

$$u_x(x_0, y_0) = v_y(x_0, y_0), \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Moreover, in this case we have $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Definition 30. The equations

$$u_x(x_0, y_0) = v_y(x_0, y_0), \quad u_y(x_0, y_0) = -v_x(x_0, y_0)$$

are called the Cauchy-Riemann equations.

Theorem 31. Let $f : D \to \mathbb{C}$ where $D$ is a subset of $\mathbb{C}$. Suppose that $z_0 = x_0 + iy_0$ is an interior point of $D$. Suppose that $f(x + iy) = u(x,y) + iv(x,y)$ for all points of $D$. Suppose that the first partial derivatives of $u$ and $v$ exist in an $\epsilon$ neighborhood of $(x_0, y_0)$ and are continuous at $(x_0, y_0)$. Finally, suppose that the Cauchy-Riemann equations

$$u_x(x_0, y_0) = v_y(x_0, y_0), \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

are satisfied. Then $f'(z_0)$ exists, and $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Theorem 32. (Polar form of Cauchy-Riemann equations). Let $f : D \to \mathbb{C}$ where $D$ is a subset of $\mathbb{C}$. Suppose that $z_0 = r_0e^{i\theta_0}$ is an interior point of $D$. Suppose that for $z \in D$ with $z = re^{i\theta}$, we have $f(z) = u(r, \theta) + iv(r, \theta)$. Suppose that the first partial derivatives of $u$ and $v$ with respect to $r$ and $\theta$ exist in an $\epsilon$ neighborhood of $z_0$ and are continuous at $z_0$. Finally, suppose that the equations

$$ru_r(r_0, \theta_0) = v_\theta(r_0, \theta_0), \quad u_\theta(r_0, \theta_0) = -rv_r(r_0, \theta_0).$$

are satisfied. Then $f'(z_0)$ exists, and $f'(z_0) = e^{-i\theta_0}(u_r(r_0, \theta_0) + iv_r(r_0, \theta_0))$. 
**Definition 33.** Let \( f : D \rightarrow \mathbb{C} \) where \( D \) is a subset of \( \mathbb{C} \). If \( S \) is an open subset of \( D \), we say that \( f \) is analytic on \( S \) if and only if \( f'(z) \) exists at each \( z \in S \). If \( z_0 \) is an interior point of \( D \), we say that \( f \) is analytic at \( z_0 \) if and only if \( f \) is analytic on some neighborhood of \( z_0 \). Finally, if \( D = \mathbb{C} \) and \( f \) is analytic on \( \mathbb{C} \), we say that \( f \) is an entire function.

**Theorem 34.** Let \( f : D \rightarrow \mathbb{C} \) where \( D \) is a subset of \( \mathbb{C} \). Suppose that \( D \) is a domain (a nonempty, open, connected set). If \( f'(z) = 0 \) for all \( z \in D \), then \( f \) is constant on \( D \).

**Definition 35.** Let \( f : D \rightarrow \mathbb{C} \) where \( D \) is a subset of \( \mathbb{C} \). Suppose that \( z_0 \in D \). If every neighborhood of \( z_0 \) includes a point \( z \) such that \( f \) is analytic at \( z \), but \( f \) is not analytic at \( z_0 \), we say that \( z_0 \) is a singular point of \( f \).

**Definition 36.** Let \( h : D \rightarrow \mathbb{R} \) where \( D \) is a subset of the \( xy \) plane. We say that \( h \) is harmonic in \( D \) if and only if \( h \) has continuous partial derivatives of the first and second order and

\[
h_{xx} + h_{yy} = 0
\]

everywhere in \( D \). This equation is known as Laplace’s equation.

**Theorem 37.** Let \( f : D \rightarrow \mathbb{C} \) where \( D \) is a subset of \( \mathbb{C} \). Suppose that \( D \) is a domain (a nonempty, open, connected set). Suppose that \( f(x + iy) = u(x,y) + iv(x,y) \) for all points of \( D \). If \( f \) is analytic on \( D \), then \( u \) and \( v \) are harmonic in \( D \).

**Corollary 38.** Let \( f : D \rightarrow \mathbb{C} \) where \( D \) is a subset of \( \mathbb{C} \). Suppose that \( D \) is a domain (a nonempty, open, connected set). Suppose that \( f \) is analytic on \( D \), and also the conjugate \( f : D \rightarrow \mathbb{C} \) is analytic on \( D \). Then \( f \) is constant on \( D \).

**Corollary 39.** Let \( f : D \rightarrow \mathbb{C} \) where \( D \) is a subset of \( \mathbb{C} \). Suppose that \( D \) is a domain (a nonempty, open, connected set). Suppose that \( f \) is analytic on \( D \), and \( |f(z)| \) is constant on \( D \). Then \( f \) is constant on \( D \).

**Definition 40.** For any complex number, \( z = x + iy \) we define \( e^z = e^x e^{iy} \). We sometimes write \( \exp(z) \) for \( e^z \).

**Proposition 41.** The function \( e^z \) is entire, and \( \frac{d}{dz} e^z = e^z \) for all complex numbers \( z \).

**Proposition 42.** Let \( z \) and \( w \) be complex numbers. We have:

1. \( e^{z+w} = e^z e^w \).
2. \( e^{z+2\pi i} = e^z \).
3. \( e^{z-w} = \frac{e^z}{e^w} \).
4. \( (e^z)^n = e^{nz} \) for any positive integer \( n \).

**Proposition 43.** Let \( z = x + iy \) and \( w = re^{i\theta} \neq 0 \). Then \( e^z = w \) if and only if

\[
z = \ln r + i(\theta + 2n\pi)
\]

for some integer \( n \).

**Proposition 44.** Let \( A \) denote the set of complex numbers \( x + iy \) with \( -\pi < y \leq \pi \). Let \( B \) denote the set of non-zero complex numbers. Let \( h : A \rightarrow B \) be defined by \( h(z) = e^z \). Let \( g : B \rightarrow A \) be defined by \( g(w) = \ln |w| + i \text{Arg}(w) \). Then \( h \) and \( g \) are inverse functions of each other.

**Definition 45.** For any non-zero complex number \( w \) we define \( \text{Log}(w) = \ln |w| + i \text{Arg}(w) \) and \( \log(w) = \ln |w| + i \text{arg}(w) \). Also, we say that \( \text{Log}(w) \) is the principal value of \( \log(w) \). Note that \( \log(w) \) is a multiple-valued function.

**Proposition 46.** For any non-zero complex number \( w \), we have

\[
\log(w) = \{ z \in \mathbb{C} : e^z = w \}.
\]

**Proposition 47.** \( \frac{d}{dz} \text{Log}(z) = \frac{1}{z} \) for all \( z \neq 0 \) with \( -\pi < \text{Arg}(z) < \pi \).

**Definition 48.** For any non-zero complex number \( z \) and any complex number \( c \) we define \( z^c = \exp(c \log z) \). Also, the principal value of \( z^c \) is defined to be \( \exp(c \text{Log} z) \).
Proposition 49. For any non-zero complex number \( z \) and any positive integer \( n \), the set \( z^{\frac{1}{n}} \) is coincides with the set of \( n \)-th roots of \( z \).

Proposition 50. Fix a complex number \( c \), and fix a branch of \( \log(z) \). Let \( f(z) = z^c \) denote the (single-valued) function attained by using this branch in the definition. Then \( f \) is analytic on the domain given by \( z \neq 0 \) and \( \alpha < \arg(z) < \alpha + 2\pi \). Moreover \( f'(z) = cz^{c-1} \).

Proposition 51. Fix a complex number \( c \neq 0 \), and fix one value of \( \log(c) \). Let \( f(z) = c^z \) denote the (single-valued) function attained. Then \( f \) is entire, and \( f'(z) = c^z \log(c) \).

Definition 52. The sine and cosine functions are defined by

\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.
\]

Proposition 53. The sine and cosine functions are entire, and

\[
\frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \cos z = -\sin z.
\]

Remark 54. The 4 other trigonometric functions of a complex variable can be defined in terms of the sine and cosine functions. Moreover, the standard trigonometric identities for a real variable continue to hold for a complex variable.

Proposition 55. For any complex number \( z = x + iy \),

\[
\sin z = \sin x \cosh(y) + i \cos x \sinh(y) \quad \cos z = \cos x \cosh(y) - i \sin x \sinh(y).
\]

Definition 56. The hyperbolic sine and cosine functions are defined by

\[
\sinh(z) = \frac{e^z - e^{-z}}{2}, \quad \cosh(z) = \frac{e^z + e^{-z}}{2}.
\]

Proposition 57. The hyperbolic sine and cosine functions are entire, and

\[
\frac{d}{dz} \sinh(z) = \cosh(z), \quad \frac{d}{dz} \cosh(z) = \sinh(z).
\]

Proposition 58. Let \( z \) be a complex number.

\[
\sinh(z) = -i \sin(iz), \quad \cosh(z) = \cos(iz).
\]

Proposition 59. For any complex number \( z = x + iy \),

\[
\sinh(z) = \sinh(x) \cos(y) + i \cosh(x) \sin(y) \quad \cosh(z) = \cosh(x) \cos(y) + i \sinh(x) \sin(y).
\]