the argument is invalid. Translating the formula \( L \lor S \) back into English, we see that the first premise could have been stated more simply as “John is either lazy or stupid (or both).” But from this premise and the second premise (that John is stupid), it clearly doesn’t follow that he’s not lazy, because he might be both stupid and lazy.

**Example 1.2.4.** Which of these formulas are equivalent?

\[ \neg (P \land Q), \quad \neg P \land \neg Q, \quad \neg P \lor \neg Q. \]

**Solution**

Here’s a truth table for all three statements. (You should check it yourself!)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( \neg (P \land Q) )</th>
<th>( \neg P \land \neg Q )</th>
<th>( \neg P \lor \neg Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

The third and fifth columns in this table are identical, but they are different from the fourth column. Therefore, the formulas \( \neg (P \land Q) \) and \( \neg P \lor \neg Q \) are equivalent, but neither is equivalent to the formula \( \neg P \land \neg Q \). This should make sense if you think about what all the symbols mean. For example, suppose \( P \) stands for the statement “The Yankees won last night” and \( Q \) stands for “The Red Sox won last night.” Then \( \neg (P \land Q) \) would mean “The Yankees and the Red Sox did not both win last night,” and \( \neg P \lor \neg Q \) would mean “Either the Yankees or the Red Sox lost last night”; these statements clearly convey the same information. On the other hand, \( \neg P \land \neg Q \) would mean “The Yankees and the Red Sox both lost last night,” which is an entirely different statement.

You can check for yourself by making a truth table that the formula \( \neg P \land \neg Q \) from Example 1.2.4 is equivalent to the formula \( \neg (P \lor Q) \). (To see that this equivalence makes sense, notice that the statements “Both the Yankees and the Red Sox lost last night” and “Neither the Yankees nor the Red Sox won last night” mean the same thing.) This equivalence and the one discovered in Example 1.2.4 are called **DeMorgan’s laws.**

In analyzing deductive arguments and the statements that occur in them it is helpful to be familiar with a number of equivalences that come up often. Verify the equivalences in the following list yourself by making truth tables, and check that they make sense by translating the formulas into English, as we did in Example 1.2.4.
DeMorgan’s laws
\[-(P \land Q)\text{ is equivalent to } \neg P \lor \neg Q.\]
\[-(P \lor Q)\text{ is equivalent to } \neg P \land \neg Q.\]

Commutative laws
\[P \land Q\text{ is equivalent to } Q \land P.\]
\[P \lor Q\text{ is equivalent to } Q \lor P.\]

Associative laws
\[P \land (Q \land R)\text{ is equivalent to } (P \land Q) \land R.\]
\[P \lor (Q \lor R)\text{ is equivalent to } (P \lor Q) \lor R.\]

Idempotent laws
\[P \land P\text{ is equivalent to } P.\]
\[P \lor P\text{ is equivalent to } P.\]

Distributive laws
\[P \land (Q \lor R)\text{ is equivalent to } (P \land Q) \lor (P \land R).\]
\[P \lor (Q \land R)\text{ is equivalent to } (P \lor Q) \land (P \lor R).\]

Absorption laws
\[P \lor (P \land Q)\text{ is equivalent to } P.\]
\[P \land (P \lor Q)\text{ is equivalent to } P.\]

Double Negation law
\[\neg \neg P\text{ is equivalent to } P.\]

Notice that because of the associative laws we can leave out parentheses in formulas of the forms \(P \land Q \land R\) and \(P \lor Q \lor R\) without worrying that the resulting formula will be ambiguous, because the two possible ways of filling in the parentheses lead to equivalent formulas.

Many of the equivalences in the list should remind you of similar rules involving \(+,\ ,\ ,\ \text{and } -\) in algebra. As in algebra, these rules can be applied to more complex formulas, and they can be combined to work out more complicated equivalences. Any of the letters in these equivalences can be replaced by more complicated formulas, and the resulting equivalence will still be true. For example, by replacing \(P\) in the double negation law with the formula \(Q \lor \neg R\), you can see that \(\neg \neg (Q \lor \neg R)\) is equivalent to \(Q \lor \neg R\). Also, if two formulas are equivalent, you can always substitute one for the other in any expression and the results will be equivalent. For example, since \(\neg \neg P\) is equivalent to
From the truth table it is clear that the first formula is a tautology, the second a contradiction, and the third neither. In fact, since the last column is identical to the first, the third formula is equivalent to \( P \).

We can now state a few more useful laws involving tautologies and contradictions. You should be able to convince yourself that all of these laws are correct by thinking about what the truth tables for the statements involved would look like.

**Tautology laws**

\[
P \land (\text{a tautology}) \text{ is equivalent to } P.
\]

\[
P \lor (\text{a tautology}) \text{ is a tautology.}
\]

\[
\neg (\text{a tautology}) \text{ is a contradiction.}
\]

**Contradiction laws**

\[
P \land (\text{a contradiction}) \text{ is a contradiction.}
\]

\[
P \lor (\text{a contradiction}) \text{ is equivalent to } P.
\]

\[
\neg (\text{a contradiction}) \text{ is a tautology.}
\]

**Example 1.2.7.** Find simpler formulas equivalent to these formulas:

1. \( P \lor (Q \land \neg P) \).
2. \( \neg (P \lor (Q \land \neg R)) \land Q \).

**Solutions**

1. \( P \lor (Q \land \neg P) \)
   
   is equivalent to \( (P \lor Q) \land (P \lor \neg P) \)
   
   (distributive law),

   which is equivalent to \( P \lor Q \)
   
   (tautology law).

   The last step uses the fact that \( P \lor \neg P \) is a tautology.

2. \( \neg (P \lor (Q \land \neg R)) \land Q \)
   
   is equivalent to \( \neg (P \land (Q \land \neg R)) \land Q \)
   
   (DeMorgan’s law),

   which is equivalent to \( (\neg P \lor \neg (Q \land \neg R)) \land Q \)
   
   (DeMorgan’s law),

   which is equivalent to \( \neg P \land (\neg Q \lor R) \land Q \)
   
   (double negation law),

   which is equivalent to \( \neg P \land ((Q \land \neg Q) \lor R) \land Q \)
   
   (associative law),

   which is equivalent to \( \neg P \land (Q \land \neg Q) \lor (Q \land R) \)
   
   (commutative law),

   which is equivalent to \( \neg P \land (Q \land R) \)
   
   (distributive law),

   which is equivalent to \( \neg P \land \neg Q \lor Q \land R \)
   
   (contradiction law).

The last step uses the fact that \( Q \land \neg Q \) is a contradiction. Finally, by the associative law for \( \land \) we can remove the parentheses without making the formula ambiguous, so the original formula is equivalent to the formula \( \neg P \land \neg Q \).
Exercises

1. Make truth tables for the following formulas:
   (a) \( \neg P \lor Q \).
   (b) \( (S \lor G) \land (\neg S \lor \neg G) \).

2. Make truth tables for the following formulas:
   (a) \( \neg(P \land (Q \lor \neg P)) \).
   (b) \( (P \lor Q) \land (\neg P \lor R) \).

3. In this exercise we will use the symbol + to mean exclusive or. In other words, \( P + Q \) means “\( P \) or \( Q \), but not both.”
   (a) Make a truth table for \( P + Q \).
   (b) Find a formula using only the connectives \( \land, \lor, \) and \( \neg \) that is equivalent to \( P + Q \). Justify your answer with a truth table.

4. Find a formula using only the connectives \( \land \) and \( \neg \) that is equivalent to \( P \lor Q \). Justify your answer with a truth table.

5. Some mathematicians use the symbol \( \downarrow \) to mean nor. In other words, \( P \downarrow Q \) means “neither \( P \) nor \( Q \)”
   (a) Make a truth table for \( P \downarrow Q \).
   (b) Find a formula using only the connectives \( \land, \lor, \) and \( \neg \) that is equivalent to \( P \downarrow Q \).
   (c) Find formulas using only the connective \( \downarrow \) that are equivalent to \( \neg P, P \lor Q, \) and \( P \land Q \).

6. Some mathematicians write \( P \mid Q \) to mean “\( P \) and \( Q \) are not both true.” (This connective is called nand, and is used in the study of circuits in computer science.)
   (a) Make a truth table for \( P \mid Q \).
   (b) Find a formula using only the connectives \( \land, \lor, \) and \( \neg \) that is equivalent to \( P \mid Q \).
   (c) Find formulas using only the connective \( \mid \) that are equivalent to \( \neg P, P \lor Q, \) and \( P \land Q \).

7. Use truth tables to determine whether or not the arguments in exercise 7 of Section 1.1 are valid.

8. Use truth tables to determine which of the following formulas are equivalent to each other:
   (a) \( (P \land Q) \lor (\neg P \land \neg Q) \).
   (b) \( \neg P \lor Q \).
   (c) \( (P \lor \neg Q) \land (Q \lor \neg P) \).
   (d) \( \neg(P \lor Q) \).
   (e) \( (Q \land P) \lor \neg P \).

9. Use truth tables to determine which of these statements are tautologies, which are contradictions, and which are neither:
Truth Tables

(a) \((P \lor Q) \land (\neg P \lor \neg Q)\).
(b) \((P \lor Q) \land (\neg P \land \neg Q)\).
(c) \((P \lor Q) \lor (\neg P \lor \neg Q)\).
(d) \([P \land (Q \lor \neg R)] \lor (\neg P \lor R)\).

10. Use truth tables to check these laws:
   (a) The second DeMorgan's law. (The first was checked in the text.)
   (b) The distributive laws.

*11. Use the laws stated in the text to find simpler formulas equivalent to these formulas. (See Examples 1.2.5 and 1.2.7.)
   (a) \(\neg (\neg P \land \neg Q)\).
   (b) \((P \land Q) \lor (P \land \neg Q)\).
   (c) \(\neg (P \land \neg Q) \lor (\neg P \land Q)\).

12. Use the laws stated in the text to find simpler formulas equivalent to these formulas. (See Examples 1.2.5 and 1.2.7.)
   (a) \(\neg (\neg P \lor Q) \lor (P \land \neg R)\).
   (b) \(\neg (\neg P \land Q) \lor (P \land \neg R)\).
   (c) \((P \land R) \lor [\neg R \land (P \lor Q)]\).

13. Use the first DeMorgan's law and the double negation law to derive the second DeMorgan's law.

*14. Note that the associative laws say only that parentheses are unnecessary when combining three statements with \(\land\) or \(\lor\). In fact, these laws can be used to justify leaving parentheses out when more than three statements are combined. Use associative laws to show that \([P \land (Q \land R)] \land S\) is equivalent to \((P \land Q) \land (R \land S)\).

15. How many lines will there be in the truth table for a statement containing \(n\) letters?

*16. Find a formula involving the connectives \(\land\), \(\lor\), and \(\neg\) that has the following truth table:

\[
\begin{array}{ccc}
P & Q & ??? \\
F & F & T \\
F & T & F \\
T & F & T \\
T & T & T \\
\end{array}
\]

17. Find a formula involving the connectives \(\land\), \(\lor\), and \(\neg\) that has the following truth table:

\[
\begin{array}{ccc}
P & Q & ??? \\
F & F & F \\
F & T & T \\
T & F & T \\
T & T & F \\
\end{array}
\]
18. Suppose the conclusion of an argument is a tautology. What can you conclude about the validity of the argument? What if the conclusion is a contradiction? What if one of the premises is either a tautology or a contradiction?

1.3. Variables and Sets

In mathematical reasoning it is often necessary to make statements about objects that are represented by letters called variables. For example, if the variable \( x \) is used to stand for a number in some problem, we might be interested in the statement “\( x \) is a prime number.” Although we may sometimes use a single letter, say \( P \), to stand for this statement, at other times we will revise this notation slightly and write \( P(x) \), to stress that this is a statement about \( x \). The latter notation makes it easy to talk about substituting some number for \( x \) in the statement. For example, \( P(7) \) would represent the statement “\( 7 \) is a prime number,” and \( P(a + b) \) would mean “\( a + b \) is a prime number.” If a statement contains more than one variable, our abbreviation for the statement will include a list of all the variables involved. For example, we might represent the statement “\( p \) is divisible by \( q \)” by \( D(p, q) \). In this case, \( D(12, 4) \) would mean “\( 12 \) is divisible by \( 4 \).”

Although you have probably seen variables used most often to stand for numbers, they can stand for anything at all. For example, we could let \( M(x) \) stand for the statement “\( x \) is a man,” and \( W(x) \) for “\( x \) is a woman.” In this case, we are using the variable \( x \) to stand for a person. A statement might even contain several variables that stand for different kinds of objects. For example, in the statement “\( x \) has \( y \) children,” the variable \( x \) stands for a person, and \( y \) stands for a number.

Statements involving variables can be combined using connectives, just like statements without variables.

Example 1.3.1. Analyze the logical forms of the following statements:

1. \( x \) is a prime number, and either \( y \) or \( z \) is divisible by \( x \).
2. \( x \) is a man and \( y \) is a woman and \( x \) likes \( y \), but \( y \) doesn’t like \( x \).

Solutions

1. We could let \( P \) stand for the statement “\( x \) is a prime number,” \( D \) for “\( y \) is divisible by \( x \),” and \( E \) for “\( z \) is divisible by \( x \).” The entire statement would then be represented by the formula \( P \land (D \lor E) \). But this analysis, though not incorrect, fails to capture the relationship between the statements.