This pattern is especially evident on the right of Figure 3.3, where each $\binom{n}{k}$ is worked out. Notice how 21 is the sum of the numbers 6 and 15 above it. Similarly, 5 is the sum of the 1 and 4 above *it* and so on.

This arrangement is called **Pascal's triangle**, after Blaise Pascal, 1623–1662, a French philosopher and mathematician who discovered many of its properties. We've shown only the first eight rows, but the triangle extends downward forever. We can always add a new row at the bottom by placing a 1 at each end and obtaining each remaining number by adding the two numbers above its position. Doing this in Figure 3.3 (right) gives a new bottom row

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1.$$

This row consists of the numbers $\binom{8}{k}$ for $0 \le k \le 8$, and we have computed them without the formula $\binom{8}{k} = \frac{8!}{k!(8-k)!}$. Any $\binom{n}{k}$ can be computed this way.

The very top row (containing only 1) of Pascal's triangle is called *Row* 0. Row 1 is the next down, followed by Row 2, then Row 3, etc. Thus Row n lists the numbers $\binom{n}{k}$ for $0 \le k \le n$. Exercises 3.5.13 and 3.5.14 established

$$\binom{n}{k} = \binom{n}{n-k}, \tag{3.4}$$

for each $0 \le k \le n$. In words, the kth entry of Row n of Pascal's triangle equals the $(n-k)$th entry. This means that Pascal's triangle is symmetric with respect to the vertical line through its apex, as is evident in Figure 3.3.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\cdots & & & & & \vdots & & & & & \cdots
\end{array}
\qquad
\begin{array}{c}
1 \\
1x + 1y \\
1x^2 + 2xy + 1y^2 \\
1x^3 + 3x^2y + 3xy^2 + 1y^3 \\
1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \\
1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5 \\
\vdots
\end{array}
$$

Figure 3.4. The n^{th} row of Pascal's triangle lists the coefficients of $(x+y)^n$

Notice that Row n appears to be a list of the coefficients of $(x+y)^n$. For example $(x+y)^2 = \mathbf{1}x^2 + \mathbf{2}xy + \mathbf{1}y^2$, and Row 2 lists the coefficients 1 2 1. Also $(x+y)^3 = \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^3$, and Row 3 is 1 3 3 1. See Figure 3.4, which suggests that the numbers in Row n are the coefficients of $(x+y)^n$.