

This pattern is especially evident on the right of Figure 3.3, where each  $\binom{n}{k}$  is worked out. Notice how 21 is the sum of the numbers 6 and 15 above it. Similarly, 5 is the sum of the 1 and 4 above it and so on.

This arrangement is called **Pascal's triangle**, after Blaise Pascal, 1623–1662, a French philosopher and mathematician who discovered many of its properties. We've shown only the first eight rows, but the triangle extends downward forever. We can always add a new row at the bottom by placing a 1 at each end and obtaining each remaining number by adding the two numbers above its position. Doing this in Figure 3.3 (right) gives a new bottom row

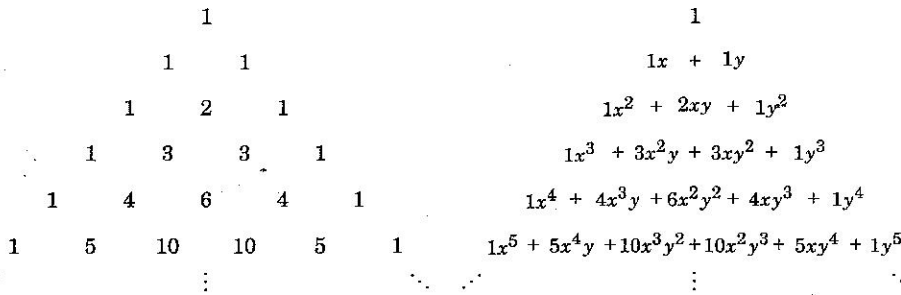
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1.$$

This row consists of the numbers  $\binom{8}{k}$  for  $0 \leq k \leq 8$ , and we have computed them without the formula  $\binom{8}{k} = \frac{8!}{k!(8-k)!}$ . Any  $\binom{n}{k}$  can be computed this way.

The very top row (containing only 1) of Pascal's triangle is called *Row 0*. Row 1 is the next down, followed by Row 2, then Row 3, etc. Thus Row  $n$  lists the numbers  $\binom{n}{k}$  for  $0 \leq k \leq n$ . Exercises 3.5.13 and 3.5.14 established

$$\binom{n}{k} = \binom{n}{n-k}, \tag{3.4}$$

for each  $0 \leq k \leq n$ . In words, the  $k$ th entry of Row  $n$  of Pascal's triangle equals the  $(n - k)$ th entry. This means that Pascal's triangle is symmetric with respect to the vertical line through its apex, as is evident in Figure 3.3.



**Figure 3.4.** The  $n^{\text{th}}$  row of Pascal's triangle lists the coefficients of  $(x + y)^n$

Notice that Row  $n$  appears to be a list of the coefficients of  $(x + y)^n$ . For example  $(x + y)^2 = 1x^2 + 2xy + 1y^2$ , and Row 2 lists the coefficients 1 2 1. Also  $(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$ , and Row 3 is 1 3 3 1. See Figure 3.4, which suggests that the numbers in Row  $n$  are the coefficients of  $(x + y)^n$ .