Sets and Logic, Dr. Block, Lecture Notes, 3-23-2020

Here is another example.

Example. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathbb{R}$ and $|x - 2| < \delta$, then $|x^2 - 9x + 14| < \epsilon$.

First, let's try to think out how we can arrive at a proof.

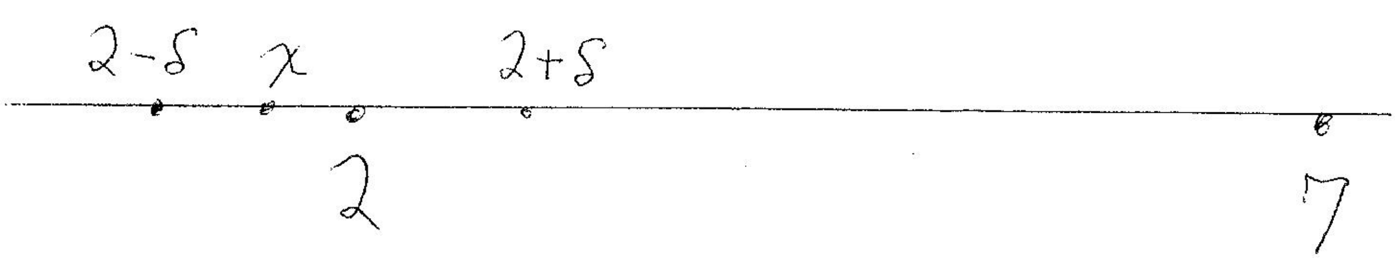
We know the first line of the proof will be: Suppose $\epsilon > 0$. We must then exhibit our choice of δ. Note that the following are equivalent:

$$|x - 2| < \delta$$

$$-\delta < x - 2 < \delta$$

$$2 - \delta < x < 2 + \delta$$

So, by choosing a small enough δ, we can get the outcome that any x satisfying the hypothesis will be close to 2 on the real line. Also, we observe that $|x^2 - 9x + 14| = |x - 2| \cdot |x - 7|$. Let's visualize the real numbers as the real line.



To figure out a δ that will work, we think about what we want to be true, namely
if $x \in \mathbb{R}$ and $|x-2| < \delta$, then $|x-2| \cdot |x-7| < \epsilon$.

We have the most control of the factor $|x-2|$, so we deal with the other factor first.

Let's make sure that $\delta \leq 1$. The choice of the positive real number 1 is somewhat arbitrary. We could just as well use 2 for example. As long as $\delta \leq 1$, we can say that if $|x-2| < \delta$, then $1 < x < 3$. So we will have

$$|x-7| = 7-x < 6.$$

So, we see that if we set

$$\delta = \min\{1, \frac{\epsilon}{6}\},$$

then we will have the desired conlusion.

Here is the proof.

Proof. Suppose that $\epsilon > 0$. Set

$$\delta = \min\{1, \frac{\epsilon}{6}\}$$

Suppose that $x \in \mathbb{R}$ and $|x-2| < \delta$. Then, as $1 < x < 3$, we have $|x-7| = 7-x < 6$. Therefore,

$$|x^2 - 9x + 14| = |x-2| \cdot |x-7| < \frac{\epsilon}{6} \cdot 6 = \epsilon.$$

□

Now, suppose we had chosen the positive real number 2 instead of the positive real number 1. So we would make sure that $\delta \leq 2$, instead of $\delta \leq 1$.

This would lead to another proof as follows:

Alternate Proof. Suppose that $\epsilon > 0$. Set

$$\delta = \min\{2, \frac{\epsilon}{7}\}$$

Suppose that $x \in \mathbb{R}$ and $|x-2| < \delta$. Then, as $0 < x < 4$, we have $|x-7| = 7-x < 7$. Therefore,

$$|x^2 - 9x + 14| = |x-2| \cdot |x-7| < \frac{\epsilon}{7} \cdot 7 = \epsilon.$$

$\square$