Sets and Logic, Dr. Block, Lecture Notes, 3-16-2020

First, we recall the definition and basic properties of the absolute value of a real number.

Definition. Let $x \in \mathbb{R}$. The absolute value of x is denoted $|x|$ and defined by:
$|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

Theorem (Properties of Absolute Value). Suppose that $x, y \in \mathbb{R}$. Then:
$|x| \geq 0$;
$|x \cdot y| = |x| \cdot |y|$;
$|x + y| \leq |x| + |y|$.
If $y > 0$ then we have the following: $|x| < y$ if and only if $-y < x < y$;

Also, it may be useful to note that if $x, y \in \mathbb{R}$, then $|x - y|$ represents the distance from x to y on the real line.

Next, we consider an example, where the absolute value appears. Note that this example is similar to the second problem on Problem Set 1.

Example. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathbb{R}$ and $|x - \sqrt{3}| < \delta$, then $|x^3 - 3x| < \epsilon$.

First, let's try to think out how we can arrive at a proof.

We know the first line of the proof will be: Suppose $\epsilon > 0$. We must then exhibit our choice of δ. Note that the following are equivalent:

$$|x - \sqrt{3}| < \delta$$

$$-\delta < x - \sqrt{3} < \delta$$

$$\sqrt{3} - \delta < x < \sqrt{3} + \delta$$

So, by choosing a small enough δ, we can get the outcome that any x satisfying the hypothesis will be close to $\sqrt{3}$ on the real line.

To figure out a δ that will work, we think about what we want to be true, namely if $x \in \mathbb{R}$ and $|x - \sqrt{3}| < \delta$, then $|x^3 - 3x| < \epsilon$.

We observe that we can factor and use one of the properties of absolute value, to see that

$$|x^3 - 3x| = |x| \cdot |x - \sqrt{3}| \cdot |x + \sqrt{3}|.$$

We now think about achieving the result that the product of these three factors is less than ϵ.

We have the most control of the factor $|x - \sqrt{3}|$, so we deal with the other two factors first.

We know that $1 < \sqrt{3} < 2$. So if x is close enough to $\sqrt{3}$, then we will have $1 < x < 3$. How close to $\sqrt{3}$ do we need x to be? The answer is we need the distance from x to $\sqrt{3}$ to be less than the mininmum of the distances of 1 and 2 to $\sqrt{3}$. So, we will plan to choose the δ to be at most $\min\{2 - \sqrt{3}, \sqrt{3} - 1\}$. Here the "min" means the minimum of the two real numbers.

Now, as long as we have $1 < x < 2$, we will also have $|x| = x < 2$, and $|x + \sqrt{3}| = x + \sqrt{3} < 4$.

Finally, assuming that we have $|x| < 2$, and $|x + \sqrt{3}| < 4$, we consider how small does $|x - \sqrt{3}|$ have to be to get the product

$$|x| \cdot |x - \sqrt{3}| \cdot |x + \sqrt{3}|$$

less than ϵ? The answer is less than $\frac{\epsilon}{8}$. So now we are ready to write the proof.

Example. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathbb{R}$ and $|x - \sqrt{3}| < \delta$, then $|x^3 - 3x| < \epsilon$.

Proof. Suppose that $\epsilon > 0$. Set $\delta = \min\{2 - \sqrt{3}, \sqrt{3} - 1, \frac{\epsilon}{8}.\}$

Suppose that $x \in \mathbb{R}$ and $|x - \sqrt{3}| < \delta$. Then

$$|x^3 - 3x| = |x| \cdot |x - \sqrt{3}| \cdot |x + \sqrt{3}| < 2 \cdot 4 \cdot \frac{\epsilon}{8} = \epsilon.$$

□

We conclude this lecture with another example of an existence and uniqueness proof.

Example. Prove that for every real number y with $0 < y < 1$ there exists a unique real number x with $x > 0$ such that

$$y = \frac{1}{x^2 + 1}.$$

Before giving the proof, let's recall in general how this type of proof is done.

To prove a statement of this form: "There exists a unique $x \in A$ such that $P(x)$ holds".

Begin the proof of existence by exhibiting a particular $x_0 \in A$.

End the proof of existence by proving that $P(x_0)$ holds.

Prove uniqueness as follows:

Suppose $x \in A$ and $P(x)$ holds. Then prove that $x = x_0$.

An alternate way to prove uniqueness is: Suppose that $x_1 \in A$, $x_2 \in A$ and both $P(x_1)$ and $P(x_2)$ hold. Then prove that $x_1 = x_2$.

In the proof that follows, note that there is some scratch work necessary to figure out the x. But this is not part of the formal proof.

Proof. Suppose that $y \in \mathbb{R}$ and $0 < y < 1$.

First, we prove existence. Set

$$x = \sqrt{\frac{1-y}{y}}.$$

Note that x is a well-defined real number as $y > 0$, and $1 - y > 0$. Also, since the square root of a positive real numbers is positive, we have $x > 0$. Now,

$$y(x^2 + 1) = y(\frac{1-y}{y} + 1) = y(\frac{1-y+y}{y}) = 1.$$

Since $y \neq 0$, we may divide by y and obtain

$$y = \frac{1}{x^2 + 1}.$$

This proves existence.

Second, we prove uniqueness. Suppose that $v, w \in R$ satisfy

$$v > 0,\ y = \frac{1}{v^2+1},\ w > 0,\ y = \frac{1}{w^2+1}.$$

Then we have

$$\frac{1}{v^2+1} = \frac{1}{w^2+1}.$$
$$v^2 + 1 = w^2 + 1$$
$$v^2 = w^2.$$

Since $v > 0$ and $w > 0$ it follows that $v + w$. This proves uniqueness.

□

Note that we have used the alternate way mentioned above to prove uniqueness. Also, we have used v and w instead of x_1 and x_2.

Feel free to email me with questions.