Sets and Logic, Dr. Block, Lecture Notes, 3-18-2020

We begin Chapter 8. In addition to carefully reading these Lecture Notes you should read Sections 8.1, 8.2, and 8.3 in the text, and work on Exercises 1 -18 at the end of Chapter 8.

You should **keep in mind the following three items** from the Chapter 8 course notes:

70. To prove that $a \in \{x : P(x)\}$, prove that $P(a)$ is true. To prove that $a \in \{x \in S : P(x)\}$, prove that $a \in S$ and prove that $P(a)$ is true.

71. Suppose that A and B are sets. The statement $A \subseteq B$ is equivalent to the conditional statement:
"If $a \in A$, then $a \in B$."
So any of the methods for proving conditional statements may be used.

72. One common method to prove that two sets are equal is to prove that each of the sets is a subset of the other.

As examples, we look at two of the exercies.

Exercise 2. Prove that $\{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.

Proof. First, we prove that

$$\{6n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}.$$

Suppose that $x \in \{6n : n \in \mathbb{Z}\}$. Then for some integer k we have $x = 6k$. Set $s = 3k$. Then $s \in \mathbb{Z}$, and $x = 2s$. It follows that $x \in \{2n : n \in \mathbb{Z}\}$.
Set $t = 2k$. Then $t \in \mathbb{Z}$, and $x = 3t$. It follows that $x \in \{3n : n \in \mathbb{Z}\}$.
It follows that $x \in (\{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\})$.
Since, $x \in \{6n : n \in \mathbb{Z}\}$ implies that $x \in (\{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\})$, we conclude that

$$\{6n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}.$$

Second, we prove that

$$\{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} \subseteq \{6n : n \in \mathbb{Z}\}.$$

Suppose that $x \in \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$. Then, for some integers c, d we have $x = 2c$ and $x = 3d$. Since $3d = 2c$, it follows that d must be even. So $d = 2y$ for some integer y. Hence, $x = 6y$. It follows that $x \in \{6n : n \in \mathbb{Z}\}$. We conclude that

$$\{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} \subseteq \{6n : n \in \mathbb{Z}\}.$$

Therefore, since each of the two sets is a subset of the other, we have

$$\{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}.$$

□

Remark. Note that in the previous proof, in describing the set

$$\{6n : n \in \mathbb{Z}\}$$

the n is a dummy variable. We could describe that same set, by using any other variable such as x or t. So

$$\{6n : n \in \mathbb{Z}\} = \{6x : x \in \mathbb{Z}\} = \{6t : t \in \mathbb{Z}\}.$$

Whatever variable is used the set is the same. It could be described as the set of all integers which are multiples of 6.

Next we look at Exercise 18. We recall the following definitions:

Suppose that A and B are sets. Then

$x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
$x \in A \cup B$ if and only if $x \in A$ or $x \in B$.
$x \in A - B$ if and only if $x \in A$ and $x \notin B$.
$(x, y) \in (A \times B)$ if and only if $x \in A$ and $y \in B$.

Exercise 18. If A, B and C are sets, then

$$A \times (B - C) = (A \times B) - (A \times C).$$

Proof. First, we prove that

$$A \times (B - C) \subseteq (A \times B) - (A \times C).$$

Suppose that $(x, y) \in A \times (B - C)$. Then $x \in A$ and $y \in (B - C)$. So $y \in B$ and $y \notin C$. Since $x \in A$ and $y \in B$, we have $(x, y) \in A \times B$. Since $y \notin C$, we have $(x, y) \notin (A \times C)$. It follows that

$$(x, y) \in (A \times B) - (A \times C).$$

We conclude that

$$A \times (B - C) \subseteq (A \times B) - (A \times C).$$

Second, we prove that

$$(A \times B) - (A \times C) \subseteq A \times (B - C).$$

Suppose that $(x, y) \in (A \times B) - (A \times C)$. Then $(x, y) \in (A \times B)$ and $(x, y) \notin (A \times C)$. It follows that $x \in A$ and $y \in B$. From $x \in A$ along with $(x, y) \notin (A \times C)$, we conclude that $y \notin C$. Thus, $y \in (B - C)$. Therefore, $(x, y) \in A \times (B - C)$.

We conclude that

$$(A \times B) - (A \times C) \subseteq A \times (B - C).$$

Therefore, since each of the two sets is a subset of the other, we have

$$A \times (B - C) = (A \times B) - (A \times C).$$

□

Remark 1. Note that we started the proof above with
"Suppose that $(x, y) \in A \times (B - C)$."
Instead we could have started as follows:
Suppose that $t \in A \times (B - C)$. Then $t = (x, y)$ for some $x \in A$ and some $y \in (B - C)$.

We recognize that any element of the Cartesian product of two sets is an ordered pair. That justifies writing the proof as above. Either way is fine.

Remark 2.
Consider this sentence from the second half of the proof above:

From $x \in A$ along with $(x, y) \notin (A \times C)$, we conclude that $y \notin C$.

Let's think about the logic behind this statement. Let P denote the statement: $x \in A$. Let Q denote the statement: $y \in C$. Then $P \wedge Q$ is equivalent to the statement $(x, y) \in (A \times C)$, according to the definition of the Cartesian product. So the statement:

From $x \in A$ along with $(x, y) \notin (A \times C)$, we conclude that $y \notin C$.

is equivalent to the statement:

$$(P \wedge \sim (P \wedge Q)) \Rightarrow \sim Q.$$

This is a tautology!

□