Sets and Logic, Dr. Block, Lecture Notes, 3-20-2020

We continue Chapter 8. In addition to carefully reading these Lecture Notes you should read Section 8.4 in the text, and work on Exercises 19 -31 at the end of Chapter 8. We begin with this definition:

Definition 73. A natural number x is perfect if and only if the sum of all of the positive divisors of x which are less than x is equal to x.

Several examples are given in the text. Note that a natural number x is perfect if and only if the sum of all of the positive divisors of x which are less than or equal to x is equal to $2x$. This is sometimes given as the definition of a perfect number. The following theorem appears in the text as Theorem 8.1 on page 167.

Theorem 74. If $A = \{2^{n-1}(2^n - 1) : n \in \mathbb{N}, \text{and}\, 2^n - 1\, \text{is prime}\}$ and $P = \{p \in \mathbb{N} : p\, \text{is perfect}\}$, then $A \subseteq P$.

You should study the proof given in the text. An important part of the proof is the assertion that the list displayed at the bottom of Page 167 which begins with 2^0 and ends with $2^{n-1}(2^n - 1)$ is a complete list of all of the positive divisors of p. This may be justified as follows: Suppose that d is a positive divisor of p. Then d has a unique prime factorization (this will be proved in Chapter 10.) Each of the primes in the prime factorization of d must also appear in the prime factorization of p. Thus, the only primes which may appear in the prime factorization of d are 2 and $2^n - 1$.

Theorem 74 establishes a connection between prime numbers of the form $2^n - 1$ and perfect numbers. We give a name to prime numbers of this form.

Definition 75. A prime number of the form $(2^n - 1)$ for some $n \in \mathbb{N}$ is called a **Mersenne prime**.

Keep in mind the following: If 2^n-1 is prime, then n is prime. However, the converse is false.

The next theorem appears in the text as Theorem 8.2 on Page 169 in the text.

Theorem 76. If $A = \{2^{n-1}(2^n - 1) : n \in \mathbb{N}, \text{and } 2^n - 1 \text{ is prime}\}$ and $E = \{p \in \mathbb{N} : p \text{ is perfect and even}\}$, then $A = E$.

See the discussion and examples of even perfect numbers on Page 170 of the text.

We conclude this lecture with another exercise from the text.

Exercise 30. Prove that $(\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z}) = \mathbb{N} \times \mathbb{N}$.

Proof. First, we prove that

$$(\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z}) \subseteq \mathbb{N} \times \mathbb{N}.$$

Suppose that $(x, y) \in (\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z})$. Then $(x, y) \in (\mathbb{Z} \times \mathbb{N})$ and $(x, y) \in (\mathbb{N} \times \mathbb{Z})$. Since $(x, y) \in (\mathbb{N} \times \mathbb{Z})$ we have $x \in \mathbb{N}$. Since $(x, y) \in (\mathbb{Z} \times \mathbb{N})$ we have $y \in \mathbb{N}$. Since $x \in \mathbb{N}$ and $y \in \mathbb{N}$ it follows that $(x, y) \in (\mathbb{N} \times \mathbb{N})$. We conclude that

$$(\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z}) \subseteq \mathbb{N} \times \mathbb{N}.$$

Second, we prove that

$$(\mathbb{N} \times \mathbb{N}) \subseteq (\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z}).$$

Suppose that $(x, y) \in (\mathbb{N} \times \mathbb{N})$. Then $x \in \mathbb{N}$ and $y \in \mathbb{N}$. As $\mathbb{N} \subseteq \mathbb{Z}$, it follows that also $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. Thus, $(x, y) \in (\mathbb{Z} \times \mathbb{N})$ and $(x, y) \in (\mathbb{N} \times \mathbb{Z})$. Therefore, $(x, y) \in (\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z})$. We conclude that

$$(\mathbb{N} \times \mathbb{N}) \subseteq (\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z}).$$

Finally, since each of the sets is a subset of the other, we have

$$(\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z}) = \mathbb{N} \times \mathbb{N}.$$

$\square$