Sets and Logic, Dr. Block, Lecture Notes, 3-25-2020

We look at Chapter 9, Disproof. Please read Chapter 9 in the text, and look at the odd numbered problems from 1 through 25. As always, it is best to attempt to solve the problem yourself, before looking at the solution in the text.

Here are some key points for this chapter.

* To disprove a statement P, prove $\sim P$.

* The most common method used to disprove a universal statement, is to give a counterexample. Note that the negation of the statement

$$\forall x \in S, P(x)$$

is the statement

$$\exists x \in S, \sim P(x).$$

* To disprove a statement with contradiction: Suppose the statement is true and then deduce a contradiction.

* To disprove a statement of the form $P(x) \Rightarrow Q(x)$, produce an example of x that makes $P(x)$ true and $Q(x)$ false.

We consider two examples taken from even numbered exercises on pages 178 and 179 in the text. In each problem we are asked to decide whether the statement is true or false. If the statement is true, we are asked to prove it. If the statement is false, we are asked to disprove it.

A great deal of mathematical research involves proving or disproving a conjecture. This often involves going back and forth between trying to prove the conjecture and trying to find a counterexample.

Here is an example.

Exercise 26. If $A = B - C$, then $B = A \cup C$.

Let's try to prove the statement first.

Suppose $A = B - C$. We want to prove that $B = A \cup C$. To prove that these two sets are equal, we must prove that each set is a subset of the other. Let's try to prove that $A \cup C \subseteq B$.

Suppose that $x \in A \cup C$. Then either $x \in A$ or $x \in C$. We must consider each case.

Case 1. $x \in A$. Then since $A = B - C$, we have $x \in B - C$. Therefore, $x \in B$. This is what we wanted to prove.

Case 2. $x \in C$. Using $A = B - C$, can we prove that $x \in B$? No. We can not rule out the possiblity that $x \notin B$. This leads us to construct a counerexample. We just need to have an element of C which is not an element of B. In the formal proof, we just give one counterexample.

Proof for Exercise 26. (disproof) Set $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{3, 4\}$. Then $A = \{1, 2\} = B - C$. However, $A \cup C = \{1, 2, 3, 4\}$. So, $B \neq A \cup C$.

Of course, in the previous problem, you could have looked at a few examples first. You might have found a counterexample that way. But the approach above led us to a counterexample.

Here is another example:

Exercise 22. If p and q are prime numbers for which $p < q$, then $2p + q^2$ is odd.

Let's think about this statement. We observe that $2p$ is even. So, $2p + q^2$ is odd if and only if q^2 is odd. Also, q^2 is odd if and only if q is odd. Can we find an example where q is not odd? We know that the only even prime number is 2. But, there is no prime number p with $p < 2$. So we see that there is no counterexample, so the statement is true. Also, we see how to prove the statement. Here is a proof.

Proof for Exercise 22. (proof) Suppose that p and q are prime numbers for which $p < q$. Since $p \geq 2$, it follows that $q \geq 3$. Now, as q is a prime number and $q \geq 3$, we know that q is odd. Hence q^2 is also odd. Finally, as $2p$ is even and q^2 is odd, we conclude that $2p + q^2$ is odd.

$\square$