Sets and Logic, Dr. Block, Lecture Notes, 3-27-2020

First, I will present my solutions to the problems on Problem Set 1, along with some remarks.

Problem 1. Suppose that t is a real number. Prove that there exists a real number w such that $\frac{w+1}{w-2} = t$ if and only if $t \neq 1$.

Proof.

Suppose that t is a real number.

First, we prove that if there exists a real number w such that $\frac{w+1}{w-2} = t$ then $t \neq 1$. By way of contradiction, suppose that there exists a real number w such that $\frac{w+1}{w-2} = t$ and $t = 1$. Then $\frac{w+1}{w-2} = 1$. It follows that $w + 1 = w - 2$ and $1 = -2$. This is a contradiction.

Next, we prove that if $t \neq 1$, then there exists a real number w such that $\frac{w+1}{w-2} = t$. Suppose that $t \neq 1$. Set

$$w = \frac{2t+1}{t-1}.$$

Then w is a well-defined real number since $t \neq 1$.

We claim that $w \neq 2$. We prove the claim by contradiction. Suppose that $w = 2$. Then $\frac{2t+1}{t-1} = 2$. It follows that $2t + 1 = 2t - 2$, and $1 = -2$. This is a contradiction. This proves the claim.

It follows that $\frac{w+1}{w-2}$ is a well-defined real number. Finally, we have

$$\frac{w+1}{w-2} = \frac{\frac{2t+1}{t-1} + 1}{\frac{2t+1}{t-1} - 2} = \frac{2t+1+t-1}{2t+1-2t+2} = \frac{3t}{3} = t.$$

$\square$

Remark 1 about this proof.

Note that we prove that $w \neq 2$, **before** writing down the fraction $\frac{w+1}{w-2}$.

Remark 2 about this proof.

The end of the proof should **not be written as follows:**

$$\frac{w+1}{w-2} = t.$$
$$\frac{\frac{2t+1}{t-1}+1}{\frac{2t+1}{t-1}-2} = t.$$
$$\frac{2t+1+t-1}{2t+1-2t+2} = t$$
$$t = t.$$

In general, we do not write down a statement in a proof, before we have proved the statement.

Problem 2. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathbb{R}$ and $|x-3| < \delta$, then $|x^2 - 5x + 6| < \epsilon$.

Proof.

Suppose that $\epsilon > 0$. Set $\delta = \min\{1, \frac{\epsilon}{2}\}$.
Suppose that $x \in \mathbb{R}$ and $|x-3| < \delta$. Then we have:

$$-\delta < x - 3 < \delta$$

$$3 - \delta < x < 3 + \delta.$$

Since $\delta \leq 1$, it follows that $2 < x < 4$. Hence, $|x-2| = x - 2 \leq 2$.
Therefore $|x^2 - 5x + 6| = |x-2| \cdot |x-3| < 2 \cdot \frac{\epsilon}{2} = \epsilon$.

$\square$

Remark about this proof. There are other valid ways to prove this.

Next, I want make a **remark which may be relevant for a problem on Problem Set 2.** Suppose that S and T are sets, and we want to prove that for some given x we have

$$x \in S \cup T.$$

The logical form of the statement is

$$(x \in S) \vee (x \in T).$$

Recall these ways to prove a statement of the form $P \vee Q$.
* Suppose $\sim P$ and prove Q.
* Suppose $\sim Q$ and prove P.
* Proceed by contradiction.
* Make cases and deal with each case separately. When you do this the cases must cover every possibility.

You only need to use one of these methods. In a given problem, you should think about which one would work best.

Finally, let's look at two more Exercises from Chapter 9.

Exercise 18. Prove or disprove.

If $a, b, c \in \mathbb{N}$, then at least one of $a - b,\ a + c,\ b - c$ is even.

Proof. We prove the statement. By way of contradiction suppose that $a, b, c \in \mathbb{N}$, and all three of the integers $a - b,\ a + c,\ b - c$ are odd. Recall that the sum of two odd integers is even, and the sum of an even integer and an odd integer is odd. It follows that the sum of three odd integers is odd. Hence,

$$a - b + a + c + b - c$$

is odd. But,

$$a - b + a + c + b - c = 2a.$$

It follows that $2a$ is odd. This is a contradiction, as we know that $2a$ must be even.

$\square$

Exercise 18. Prove or disprove.

If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$.

Let's think about the logical form of the hypothesis and conclusion. The logical form of $X \subseteq A \cup B$ is:

$$\forall x\,(x \in X \Rightarrow ((x \in A) \vee (x \in B))).$$

Te logical form of $X \subseteq A$ or $X \subseteq B$ is

$$(\forall x\,((x \in X) \Rightarrow (x \in A))) \vee (\forall x\,((x \in X) \Rightarrow (x \in A))).$$

Observe that the second statement does not follow from the first. The first statement allows the possilibility that some $x \in X$ are in A but not in B, while other $x \in X$ are in B but not in A. The second statement rules out this possibility. In the formal proof we just give one counterexample.

Proof. We disprove the statement. Set $X = \{1, 2\}$. Set $A = \{1\}$ and $B = \{2\}$. Then $X \subseteq A \cup B$, while X is not a subset of A and X is not a subset of B.

$\square$