Sets and Logic, Dr. Block, Lecture Notes, 4-1-2020

First, I will make some comments about the two problems on Problem Set 2. Let's look at the first problem.

1. Suppose that A, B, and C are sets. Prove that

$$(A \cup C) \subseteq (B \cup C)$$

if and only if

$$(A - C) \subseteq (B - C).$$

First note that we are asked to prove a statement of the form $P \Leftrightarrow Q$. Recall that the most common way to do this is to prove that both $P \Rightarrow Q$ and $Q \Rightarrow P$. It is best to put each of these statements with the proof in a separate paragraph. We sometimes call the two statements the two directions of the proof. Each direction may be proved by direct proof, contrapositive proof, or proof by contradiction. You do not have to use the same method for each direction.

Second, note that in each direction of the proof, we have to prove something of the form:

$$S \subseteq T$$

The logical form of this is:

$$(x \in S) \Rightarrow (x \in T)$$

So the first line of the proof should be
"Suppose $(x \in S)$."
The end of the proof should be
"$x \in T$.

Finally, in one direction of the proof, the T in the last line is $B \cup C$. So we have to prove $x \in B \cup C$. The logical form of this is

$$(x \in B) \vee (x \in C).$$

So proving a statement of the form $P \vee Q$ is involved.

Recall the ways to prove a statement of the form $P \vee Q$.

* Suppose $\sim P$ and prove Q.
* Suppose $\sim Q$ and prove P.
* Proceed by contradiction.
* Make cases and deal with each case separately. When you do this the cases must cover every possibility.

Now let's look at the second problem.

2. Prove or disprove the following:

If $a, b, c \in \mathbb{Z}$, then at least one of the three integers

$$b^2 - c^3, \ a^2 + b, \ c - a$$

is even.

Assuming that no one has found a counterexample, let's think about how we might prove the statement.

Let P denote the statement: $b^2 - c^3$ is even.
Let Q denote the statement: $a^2 + b$ is even.
Let R denote the statement: $c - a$ is even.
Then the statement we want to prove has the form:

$$P \vee Q \vee R$$

Here are some ways that we could proceed:

* Suppose $\sim P$ and $\sim Q$, and prove R.
* Suppose $\sim P$ and and $\sim R$, and prove Q.
* Suppose $\sim Q$ and and $\sim R$, and prove P.
* Proceed by contradiction.
* Make cases and deal with each case separately. When you do this the cases must cover every possibility.

Making cases will certainly work. We are given three integers a, b, and c. We know that each of these three integers is either even or odd. This leads to eight possible cases. The eight cases cover all of the possibilities. For example, one of the cases is: a is odd, b is even, and c is odd. So we could show that in each of the eiqht cases at least one of P, Q, or R holds. This is a valid method of proof.

Another approach is to proceed by contradiction. So the proof would begin:

Suppose that $a, b, c \in \mathbb{Z}$. Proceeding by contradiction, suppose that all three of the integers

$$b^2 - c^3,\ a^2 + b,\ c - a$$

are odd.

So now we have quite a bit of information. Still, we could look at cases to help get a contradiction. The cases must cover every possibility and we must get a contradiction in each case. One way we might consider is:

Case 1: a is even.

Case 2: a is odd.

$\square$