Sets and Logic, Dr. Block, Lecture Notes, 4-10-2020

We continue discussing equivalence relations. Please read Section 11.4 of the text and work on Exercises 1, 3, and 5 on page 217 of the text.

First, an important remark.

Suppose that R is an equivalence relation on A. Suppose that $x \in A$. Recall that the equivalence class of x is denoted $[x]$. In the lecture notes, we defined $[x]$ as follows:

$$[x] = \{y \in A : xRy\}.$$

We could have just as well defined $[x]$ in the following way which corresponds to the definition in the text:

$$[x] = \{y \in A : yRx\}.$$

These are equivalent, because R is symmetric. So, for $x, y \in A$ we have

$$xRy \Leftrightarrow yRx$$

So, the two sets $\{y \in A : xRy\}$ and $\{y \in A : yRx\}$ are equal. So, we can take either of the two sets to be the definition of $[x]$.

$\square$

Next, we give a proof of the theorem mentioned at the end of the lecture notes from 4-8-2020.

Theorem. Suppose that R is an equivalence relation on A. Suppose that $v, w \in A$. If $w \in [v]$, then $[w] = [v]$.

Proof. Suppose that R is an equivalence relation on A. Suppose that $v, w \in A$ and $w \in [v]$.

First, we prove that $[w] \subseteq [v]$. Suppose that $x \in [w]$. Then by definition we have wRx. Similarly, as $w \in [v]$, we also have vRw. Since R is transitive, it follows that vRx. Hence, by definition, we have $x \in [v]$. Therefore, $[w] \subseteq [v]$.

Second, we prove that $[v] \subseteq [w]$. Suppose that $x \in [v]$. Then by definition we have xRv. Moreover, as $w \in [v]$, we have vRw. Since R is transitive, it follows that xRw. Hence, by definition, we have $x \in [w]$. Therefore, $[v] \subseteq [w]$.

Since each set is a subset of the other, we conclude that $[w] = [v]$.

□

The next theorem appears in the text as Theorem 11.1 on page 215. Please study the proof given in the text carefully.

Theorem 11.1 Suppose that R is an equivalence relation on A. Suppose that $a, b \in A$. Then $[a] = [b]$ if and only if aRb.

□

We now have two definitions.

Definition. Suppose that D and E are sets. We say that D and E are **disjoint** if and only if $D \cap E = \phi$.

Definition. Suppose that $\mathcal{F}$ is a set, and each element of $\mathcal{F}$ is a set. We say that $\mathcal{F}$ is **pairwise disjoint** if and only if every pair of distinct elements of $\mathcal{F}$ are disjoint.

Definition. Suppose that A is a set and $\mathcal{F}$ is a set. We say that $\mathcal{F}$ is a partition of A if and only if each of the following 4 conditions is satisfied:

(1) $\mathcal{F} \subseteq \mathscr{P}(A)$.

(2) $\mathcal{F}$ is pairwise disjoint.

(3) $\phi \notin \mathcal{F}$

(4) The union of all of the sets which are elements of $\mathcal{F}$ is equal to A.

Let's clarify the meaning of statement (4). The union all of the sets which are elements of $\mathcal{F}$ means the set D given by:

$x \in D$ if and only if there exists some $B \in \mathcal{F}$ such that $x \in B$. With this understanding condition (4) is equivalent to the statement $D = A$.

Compare this definition to the definition of partition given in the text on page 216. Note that in the above definition the hypothesis $\mathcal{F} \subseteq \mathscr{P}(A)$ could be expresses informally as

$\mathcal{F}$ is a set, and each element of $\mathcal{F}$ is a subset of A

So, this definition agrees with the definition in the text.

Also, note that in the definition above, Condition (4) can be replaced by the following:

(5) A is a subset of the union of all of the sets which are elements of $\mathcal{F}$.

Why is this the case? Let D denote the union of all of the sets which are elements of $\mathcal{F}$. Condition (4) states that $D = A$. Condition (5) states that $A \subseteq D$. So, if we think of D and A as arbitary sets, statements (4) and (5) are not equivalent. But, in the context of this definition, the two sets are not arbitrary. We are supposing that the conditions (1), (2), and (3) hold. Condition (1) says that each element of $\mathcal{F}$ is an element of the power set of A, or equivalently, each element of $\mathcal{F}$ is a subset of A. This imples that $D \subseteq A$. So, since we know already that $D \subseteq A$, the statements $A \subseteq D$ and $A = D$ are equivalent.

If we write down the logical form of statement (5), we get an equivalent statement, which we call (6) as follows:

(6) Every element of A is an element of some set B with $B \in \mathcal{F}$.

The following proposition gives another way of expressing the definition of a partition.

Propostion. Suppose that A is a set, and $\mathcal{F} \subseteq \mathscr{P}(A)$ such that $\phi \notin \mathcal{F}$. Then $\mathcal{F}$ is a partition of A if and only if for every $x \in A$ there is a unique $B \in \mathcal{F}$ such that $x \in B$.

Why is the proposition true? Note that conditions (1) and (3) are included in the hypothesis of the Proposition. Consider that the statement:

for every $x \in A$ there is a unique $B \in \mathcal{F}$ such that $x \in B$.

The existence part of this statement is equivalent to statement (6) above. The uniqueness part of this statement is equivalent to statement (2) above.

□

The following theorem appears in the text as Theorem 11.2.

Theorem. Suppose that R is an equivalence relation on A. Then the set $\{[a] : a \in A\}$ of equivalence classes forms a partition of A.

Proof. Suppose that R is an equivalence relation on A. Let $\mathcal{F}$ denote the set of equivalence classes. We prove that conditions (1), (2), (3), and (5) are satisfied.

(1) We prove that $\mathcal{F} \subseteq \mathscr{P}(A)$. Suppose that $W \in \mathcal{F}$. Then, for some $a \in A$ we have

$$W = [a] = \{y \in A : aRy\}.$$

It follows that $W \subseteq A$, and thus, $W \in \mathscr{P}(A)$. Therefore, $\mathcal{F} \subseteq \mathscr{P}(A)$.

(2) We prove that $\mathcal{F}$ is pairwise disjoint. Suppose that D and E are elements of $\mathcal{F}$. Then, for some $d, e \in A$, we have $D = [d]$ and $E = [e]$. We must prove that if $D \neq E$, then $D \cap E = \phi$. We use contrapositive proof. Suppose that $D \cap E \neq \phi$. There is some $x \in (D \cap E)$. It follows that dRx and xRe. Since R is transitive we have dRe. It follows from Theorem 11.1 that

$$D = [d] = [e] = E.$$

Therefore, if $D \neq E$, then $D \cap E = \phi$. We conclude that $\mathcal{F}$ is pairwise disjoint.

(3) We prove that $\phi \notin \mathcal{F}$. Proceeding by contradiction, suppose that $\phi \in \mathcal{F}$. Then there is some $a \in A$ with $\phi = [a]$. Since R is reflexive, we have aRa. It follows that $a \in [a]$. Thus, we have $a \in \phi$. This is a contradiction. We conclude that $\phi \notin \mathcal{F}$.

(5) We prove that A is a subset of the union of all of the sets which are elements of $\mathcal{F}$. Suppose that $x \in A$. Set $B = [x]$. Then $x \in B$, and $B \in \mathcal{F}$. It follows that x is an element of the union of all of the sets which are elements of $\mathcal{F}$. We conclude that A is a subset of the union of all of the sets which are elements of $\mathcal{F}$.

□

You should compare this proof to the proof in the text.

□