Sets and Logic, Dr. Block, Lecture Notes, 4-15-2020

We discuss Sections 11.6, 12.1, and 12.2 in the text. Please read these Sections 11.6 and 12.1 and work on Exercises 1, 3, 5, 7, and 9 on page 228. Please also read the first part of Section 1.2 through Page 230.

In the first 5 sections of Chapter 11, we discussed relations of a set A. More generally, we can consider relations from a set A to a set B. Here is the definition.

Definition. We say that f is a relation from A to B if and only if $f \subseteq (A \times B)$.

We will mainly be interested in a special type of relation from A to B called a function. Here is the definition.

Definition. Suppose that f is a relation from A to B. We say that f is a **function** from A to B if and only if for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$. We use the notation $f : A \to B$ to indicate that f is a function from A to B. Also, if $a \in A$, we let $f(a)$ denote the unique $b \in B$ such that $(a, b) \in f$.

Problem 1. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Which of the following relations from A to B is a function?

(a) $\{(1, 4), (2, 5)\}$
(b) $\{(1, 4), (2, 5), (3, 4), (3, 5)\}$
(c) $\{(1, 4), (2, 5), (3, 4)\}$
(d) $\{(1, 4), (2, 4), (3, 4)\}$

Solution.

(a) This relation f is not a function. If we consider $a = 3$, then there does not exist $b \in B$ such that $(a, b) \in f$.

(b) This relation f is not a function. If we consider $a = 3$, then while there does exist $b \in B$ such that $(a, b) \in f$, the b is not unique.

(c) This relation f is a function and we can describe the function as follows: $f : A \to B$ given by

$$f(1) = 4,\ f(2) = 5,\ f(3) = 4.$$

(d) This relation f is a function and we can describe the function as follows: $f : A \to B$ given by

$$f(1) = 4,\ f(2) = 4,\ f(3) = 4.$$

Definition. Suppose that $f : A \to B$. The set A is called the **domain** of f. The set B is called the **codomain** or sometimes **target space** of f. The **range** of f is the set of all $b \in B$ such that there exists $a \in A$ with $f(a) = b$.

Remark. In the text a definition of equality of two functions is given. **The definition in the text is not a standard definition. Here is my definition.**

Definition. We say that two functions $f : A \to B$ and $g : C \to D$ are equal if and only if $A = C$, $B = D$, and the set f is equal to the set g.

Remark. It follows from the definition above (and also from the definition in the text) that two functions f and g from A to B are equal if and only if for every $a \in A$, $f(a) = g(a)$. So, to prove that two functions f and g from A to B are equal we would structure the proof as follows:
First line of proof: Suppose that $a \in A$.
Last line of proof: $f(a) = g(a)$.

Definition. Suppose that $f : A \to B$. We say that f is **injective** or **one-to-one** if and only if for all $a_1 \in A$ and $a_2 \in A$ if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$.

Remark. Note that the contrapositive of the statement

if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$.

is the statement

if $f(a_1) = f(a_2)$ then $a_1 = a_2$.

So either statement could be used just as well in giving the definition.

So there are two natural ways to prove that a function $f : A \to B$ is injective.

Direct approach:

First line of proof: Suppose that $a_1, a_2 \in A$ and $a_1 \neq a_2$.
Last line of proof: $f(a_1) \neq f(a_2)$.

Contrapositive approach:

First line of proof: Suppose that $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$.

Last line of proof: $a_1 = a_2$.

Suggestion: I suggest trying the contrapositive approach first.

Remark. Note that you do not have to use the variables a_1 and a_2 in writing the proof. Instead you could use s and t or v and w for example.

Here is an example.

Problem 2. Prove that the function $f : (\mathbb{R} - \{-1\}) \to \mathbb{R}$ given by

$$f(x) = \frac{2x}{x+1}$$

is injective.

Proof: Suppose that $s, t \in (\mathbb{R} - \{-1\})$ and $f(s) = f(t)$. Then

$$\frac{2s}{s+1} = \frac{2t}{t+1}.$$

$$2s(t+1) = 2t(s+1).$$

$$2st + 2s = 2ts + 2t.$$

$$2s = 2t.$$

$$s = t.$$

We conclude that f is injective.

$\square$

Definition. Suppose that $f : A \to B$. We say that f is **surjective** or **onto** B or sometimes just **onto** if and only if for every $b \in B$ there exists $a \in A$ with $f(a) = b$.

Remark. To prove that a function $f : A \to B$ is surjective, structure the proof as follows:

First line of Proof: Suppose that $b \in B$.

Next part of proof: Set $a = ???$ Deciding what to set a equal to might involve some thought or solving an equation, but this need not be included in the proof.

Last part of proof: Verify that $a \in A$, and $f(a) = b$. In some problems it is obvious that $a \in A$, so in these problems you can just state $a \in A$. In the following example this is not obvious.

Problem 3. Prove that the function $f : (\mathbb{R} - \{2\}) \to (\mathbb{R} - \{5\})$ defined by $f(x) = \frac{5x+1}{x-2}$ is surjective.

Discussion. We will start with $b \in (\mathbb{R} - \{5\})$. We need to find $a \in (\mathbb{R} - \{2\})$ with $f(a) = b$. We can try to obtain a by solving the equation

$$\frac{5x+1}{x-2} = b$$

for x. Let's do this.

$$5x + 1 = bx - 2b$$

$$(5 - b)x = -2b - 1$$

$$x = \frac{-2b - 1}{5 - b}$$

This is helpful, but what does this prove? We started with the hypothesis $\frac{5x+1}{x-2} = b$. Then we proved that $x = \frac{-2b-1}{5-b}$. So, we proved that if $\frac{5x+1}{x-2} = b$, then $x = \frac{-2b-1}{5-b}$. What we want to prove is that if $x = \frac{-2b-1}{5-b}$, then $\frac{5x+1}{x-2} = b$. **So this does not constitute a valid proof.** But, this does tell us what to set a equal to in the proof.

Note that unfortunately, **several proofs in the text that a given function is surjective are not correctly written proofs.** They are instead discussions similar to the discussion above.

Here is one way to write a valid proof.

Proof. Suppose that $b \in (\mathbb{R} - \{5\})$. Set $a = \frac{-2b-1}{5-b}$. Then a is a well-defined real number since $b \neq 5$. Moreover, $a(5 - b) = -2b - 1$. It follows that $5a - ab = -2b - 1$, so

$$5a + 1 = b(a - 2).$$

It follows from this equality that $a \neq 2$ (because if $a = 2$, then $11 = 0$). Thus, $a \in (\mathbb{R} - \{2\})$.

It also follows that

$$f(a) = \frac{5a+1}{a-2} = b.$$

We conclude that f is surjective.

□

Here is another way to write a valid proof for the same problem.

Proof. Suppose that $b \in (\mathbb{R} - \{5\})$. Set $a = \frac{-2b-1}{5-b}$. Then a is a well-defined real number since $b \neq 5$.

We claim that $a \neq 2$. We prove this claim by contradiction. Suppose that $a = 2$. Then $2 = \frac{-2b-1}{5-b}$. It follows that $10 - 2b = -2b - 1$. So, $10 = -1$. This is a contradiction. This proves the claim that $a \neq 2$. It follows that $a \in (\mathbb{R} - \{2\})$.

Finally, we have

$$f(a) = \frac{5a+1}{a-2} = \frac{5(\frac{-2b-1}{5-b}) + 1}{(\frac{-2b-1}{5-b}) - 2} = \frac{-10b - 5 + 5 - b}{-2b - 1 - 10 + 2b} = b.$$

We conclude that f is surjective.

Definition. We say that f is **bijective** if and only if f is injective and surjective.

Remark. To prove that a function is bijective, prove that the function is injective and surjective. The function f given in the Problem 3 above is bijective. We have already proved that f is surjective. Here is a proof that f is injective.

Problem 4. Prove that the function $f : (\mathbb{R} - \{2\}) \to (\mathbb{R} - \{5\})$ defined by $f(x) = \frac{5x+1}{x-2}$ is injective.

Proof. Suppose that $v, w \in (\mathbb{R} - \{2\})$ and $f(v) = f(w)$. Then

$$\frac{5v+1}{v-2} = \frac{5w+1}{w-2}.$$

It follows that

$$5vw - 10v + w - 2 = 5wv - 10w + v - 2.$$

Adding $2 - 5vw$ to each side of this equality, we obtain

$$-10v + w = -10w + v.$$

It follows that $-11v = -11w$, and hence, $v = w$.

We conclude that f is injective.

$\square$