Sets and Logic, Dr. Block, Lecture Notes, 4-20-2020

We continue discussing material from Section 12.5 of the text. Please read this section and work on Exercises 1,3, and 5 on Page 241. We begin with a lemma.

1. **Lemma.** Suppose that $f : A \to B$. Suppose also that the inverse relation f^{-1} from B to A is a function from B to A. Suppose that $x \in A$ and $y \in B$. Then $f(x) = y$ if and only if $f^{-1}(y) = x$.

   Proof. According to the definitions, the following are equivalent:

   $$f(x) = y.$$

   $$(x, y) \in f.$$

   $$(y, x) \in f^{-1}.$$

   $$f^{-1}(y) = x.$$

   □

   We have the following theorem.

2. **Theorem.** Suppose that $f : A \to B$. The inverse relation f^{-1} from B to A is a function from B to A if and only if f is bijective.

   Proof. Suppose that $f : A \to B$.

   First, we prove that if the inverse relation f^{-1} from B to A is a function from B to A, then f is bijective.

   Suppose that the inverse relation f^{-1} from B to A is a function from B to A.

   We prove that f is injective. Suppose that $v, w \in A$ and $f(v) = f(w)$. Set $y = f(v)$. Then $y \in B$, and also, $y = f(w)$. It follows from

Lemma 1 above that $f^{-1}(y) = v$ and $f^{-1}(y) = w$. Since f^{-1} is a function, it follows that $v = w$. Therefore, f is injective.

Next, we prove that f is surjective. Suppose that $b \in B$. Since f^{-1} is a function, $f^{-1}(b)$ is well-defined. Set $a = f^{-1}(b)$. Then, $a \in A$, and, by Lemma 1, $f(a) = b$. Therefore, f is surjective.

Since f is injective and surjective, it follows that f is bijective.

We conclude that if the inverse relation f^{-1} from B to A is a function from B to A, then f is bijective.

Second, we prove that if f is bijective, then the inverse relation f^{-1} from B to A is a function from B to A.

Suppose that $b \in B$. According to the definition of a function, we must prove that there exists a unique $a \in A$ such that $(b, a) \in f^{-1}$.

We prove existence first. Since f is surjective, there exists $a \in A$ with $f(a) = b$. Then, by definition, we have $(a, b) \in f$. It follows that $(b, a) \in f^{-1}$. This prove existence.

Next, we prove uniqueness. Suppose that $a_1, a_2 \in A$, $(b, a_1) \in f^{-1}$ and $(b, a_2) \in f^{-1}$. Then $(a_1, b) \in f$ and $(a_2, b) \in f$. It follows that $f(a_1) = f(a_2)$. Thus, since f is injective, we have $a_1 = a_2$. This proves uniqueness.

Since we have proved existence and uniqueness, we conclude that there exists a unique $a \in A$ such that $(b, a) \in f^{-1}$. It follows that the inverse relation f^{-1} from B to A is a function from B to A.

We conclude that if f is bijective, then the inverse relation f^{-1} from B to A is a function from B to A.

Finally, since we have proved that each statement implies the other, we conclude that the inverse relation f^{-1} from B to A is a function from B to A if and only if f is bijective.

□

Here is an example.

3. **Problem.** Let A denote the closed interval $(-\infty, -3]$. Let B denote the closed interval $[-7, \infty)$. Prove that the function $f : A \to B$ given by $f(x) = (x+3)^2 - 7$ is bijective. Find a formula for f^{-1}.

   Discussion: We will first prove that f is surjective. We will start with $b \in B$. We then must produce an $a \in A$ with $f(a) = b$. To find a we need to solve an equation as follows:

   $$(a+3)^2 - 7 = b$$

   Note that if you prefer, you could use x and y in this equation instead of a and b. We solve the equation as follows:

   $$(a+3)^2 - 7 = b$$

   $$(a+3)^2 = b+7$$

   Now, we know that there are two real numbers whose square is $b+7$, namely, $\sqrt{b+7}$ and $-\sqrt{b+7}$. Since we need to have $a \in A$, we must have $a+3 \leq 0$. So we need to choose $-\sqrt{b+7}$. So, we continue solving for a as follows:

   $$a+3 = -\sqrt{b+7}$$
   $$a = -3 - \sqrt{b+7}$$

   Now we can proceed with the formal proof:

   Proof that f is bijective.

   First, we prove that f is surjective. Suppose that $b \in B$. Set $a = -3 - \sqrt{b+7}$. Then $a \in A$, and

   $$f(a) = (a+3)^2 - 7 = ((-3 - \sqrt{b+7}) + 3)^2 - 7 = (b+7) - 7 = b.$$

   Therefore, f is surjective.

Second, we prove that f is injective. Suppose that $v, w \in A$ and $f(v) = f(w)$. Then

$$(v+3)^2 - 7 = (w+3)^2 - 7.$$

It follows that

$$(v+3)^2 = (w+3)^2.$$

Since $v, w \in A$, it follows that $v+3 \leq 0$ and $w+3 \leq 0$. From this, together with the equality above, we can conclude that $v+3 = w+3$. Thus, $v = w$. Therefore, f is injective.

Finally, since f is surjective and injective, we conclude that f is bijective.

□

Now, we know from Theorem 2, that the inverse relation f^{-1} from B to A is a function from B to A. Let's find a formula for f^{-1}. In the proof that f is surjective, we saw that if $a = -3 - \sqrt{b+7}$, then $f(a) = b$. By Lemma 1 above, it follows that $f^{-1}(b) = a$. So, a formula for if f^{-1} is given by $f^{-1}(b) = -3 - \sqrt{b+7}$. In writing this formula, the b is a dummy variable. So, for example, you could write the formula as $f^{-1}(y) = -3 - \sqrt{y+7}$ or $f^{-1}(x) = -3 - \sqrt{x+7}$.

□