Sets and Logic, Dr. Block, Lecture Notes, 4-22-2020

We finish our discussion of inverse functions. We begin with a definition.

1. **Definition.** Let A be a set. The **identity function** on A is the function $i_A : A \to A$ given by $i_A(x) = x$ for every $x \in A$.

   Recall the following lemma which we proved last time.

2. **Lemma.** Suppose that $f : A \to B$. Suppose also that the inverse relation f^{-1} from B to A is a function from B to A. Suppose that $x \in A$ and $y \in B$. Then $f(x) = y$ if and only if $f^{-1}(y) = x$.

   Also, recall the following theorem.

3. **Theorem.** Suppose that $f : A \to B$. The inverse relation f^{-1} from B to A is a function from B to A if and only if f is bijective.

   Finally, recall the following remark:

4. **Remark.** Two functions f and g from A to B are equal if and only if for every $a \in A$, $f(a) = g(a)$.

   Now we look at two new results.

5. **Theorem.** Suppose that $f : A \to B$ is bijective. Then $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

   Proof. By Theorem 3 above, f^{-1} is a function from B to A. So the compositions above are well-defined.

   First, we prove that $f^{-1} \circ f = i_A$. Note that each of the functions $f^{-1} \circ f$ and i_A has domain A and codomain A. Suppose that $x \in A$. Set $y = f(x)$. Then by Lemma 2 above, we have

   $$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x = i_A(x).$$

   It follows from Remark 4 above that $f^{-1} \circ f = i_A$.

Second, we prove that $f \circ f^{-1} = i_B$. Note that each of these two functions has domain B and codomain B. Suppose that $y \in B$. Set $x = f^{-1}(y)$. Then

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y = i_B(y).$$

Again, it follows from Remark 4 above that $f \circ f^{-1} = i_B$.

□

We also have the following theorem.

6. **Theorem:** Suppose that $f : A \to B$ and $g : B \to A$. Suppose also that $g \circ f = i_A$ and $f \circ g = i_B$. Then f and g are bijective and $g = f^{-1}$.

   We will omit the proof of this theorem. You might try to prove it yourself.

   This theorem yields a different way to prove that a function is bijective, and find the inverse function, Just present the function g and prove that each of the two compositions is the identity function on the appropriate set. Of course you have to find the function g. This would essentially be the same as finding the inverse function. But this step would not be included in the formal proof.

   Note: I will be posting solutions to the problems on Exam 2 and Problem Set 3 tomorrow.