Sets and Logic, Dr. Block, Lecture Notes, 4-3-2020

We begin Chapter 11 in the text on relations. Please study sections 11.1 and 11.2 in the text. Also try to do the odd-numbered exercises 1-11 on page 204 and 1-17 on pages 208 - 209.

Definition and Notation. A **relation** on a set A is a subset R of the Cartesian product $A \times A$. We often use the notation xRy instead of $(x, y) \in R$.

Example 1. Define a relation R on Z by

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \mid y\}.$$

We could define this same relation by stating:
R is the relation on Z satisfying xRy if and only if x divides y.

Informally, we could define the same relation as follows:
Let R be the relation divides on the set of integers.

So in this example instead of writing xRy we would write $x \mid y$.

Example 2. Suppose that we say informally: Consider the relation $<$ on the set of real numbers. What formal relation are we describing? Formally, we are describing the relation R on $\mathbb{R}$ given by

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y\}.$$

Example 3. Consider the relation R on $\mathbb{R}$ given by

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}.$$

We can think of $\mathbb{R} \times \mathbb{R}$ as the usual x, y plane, and visualize this set of ordered pairs as the circle of radius one with center at the origin in this plane.

Definition. Suppose that R is a relation on a set A.

R is **reflexive** if and only if for all $\forall x \in A, xRx$.

R is **symmetric** if and only if $\forall x, y \in A, (xRy \Rightarrow yRx)$.

R is **transitive** if and only if $\forall x, y, z \in A, ((xRy \wedge yRz) \Rightarrow xRz)$.

Fron these definitions we have the following: Suppose that R is a relation on a set A.

To prove that R is reflexive: Suppose that $x \in A$. Prove that xRx.

To prove that R is symmetric: Suppose that $x, y \in A$, and xRy. Prove that yRx.

To prove that R is transitive: Suppose that $x, y, z \in A$. Suppose also that xRy and yRz. Then prove that xRz.

Let's look again at the three examples above and consider these three properties.

Problem 1. Let R be the relation divides given in Example 1. Prove or disprove each of the following.
1. R is reflexive.
2. R is symmetric.
3. R is transitive.

Proofs.

1. We prove the statement: R is reflexive.

Suppose that $x \in \mathbb{Z}$. We know that $x = 1 \cdot x$, and also, $1 \in \mathbb{Z}$. It now follows from the definition that $x \mid x$. Therefore, R is reflexive.

2. We disprove the statement. R is symmetric.

We know that 2 and 6 are integers. Also, 2 divides 6, but 6 does not divide 2. Therefore, R is not symmetric.

3. We prove the statement. R is transitive.

Suppose that $x, y, z \in \mathbb{Z}$. Suppose also that $x \mid y$ and $y \mid z$. Then for some integers c, k we have $y = cx$ and $z = ky$. It follows that

$$z = ky = k(cx) = (kc)x.$$

Since kc is an integer, it follows that $x \mid z$. Therefore, R is transitive.

$\square$

Note that we disprove the statement: R is symmetric by giving a counterexample.

Problem 2. Let R be the relation $<$ given in Example 2. Prove or disprove each of the following.

1. R is reflexive.
2. R is symmetric.
3. R is transitive.

Proofs.

1. We disprove the statement: R is reflexive.

For the real number 1, it is not the case that $1 < 1$. So R is not reflexive.

2. We disprove the statement. R is symmetric.

We know that 1 and 2 are real numbers. Also, $1 < 2$ but it is not true that $2 < 1$. Therefore, R is not symmetric.

3. We prove the statement. R is transitive.

Suppose that $x, y, z \in \mathbb{R}$. Suppose also that $x < y$ and $y < z$. It follows from a basic order property of real numbers that $x < z$. Therefore, R is transitive.

$\square$

Note that for the two statements we disproved, there were lots of choices for the counterexample. But we only have to give one counterexample. This is because the negation of a statement of the form $\forall x \in \mathbb{R},\ P(x)$ is given by $\exists x \in \mathbb{R},\ \sim P(x)$.

Problem 3. Let R be the relation given in Example 3. Prove or disprove each of the following.

1. R is reflexive.

2. R is symmetric.
3. R is transitive.

Proofs.

1. We disprove the statement: R is reflexive.

For the real number 5, it is not the case that $5^2 + 5^2 = 1$. So $(5, 5) \notin R$. It follows that R is not reflexive.

2. We prove the statement: R is symmetric.

Suppose that $(x, y) \in R$. Then $x^2 + y^2 = 1$. So also, $y^2 + x^2 = 1$. It follows that $(y, x) \in R$. Therefore, R is symmetric.

3. We disprove the statement. R is transitive.

Set $x = 0$, $y = 1$ and $z = 0$. Since $x^2 + y^2 = 1$, we have $(x, y) \in R$. Since $y^2 + z^2 = 1$, we have $(y, z) \in R$. But, since $x^2 + z^2 = 0 \neq 1$, we have $(x, z) \notin R$. Therefore, R is not transitive.

$\square$