Sets and Logic, Dr. Block, Lecture Notes, 4-6-2020

We continue Chapter 11 in the text on relations. Please study section 11.3 in the text. Also try to do the odd-numbered exercises 1-11 on page 214.

1. **Definition.** Suppose that R is a relation on a set A. We say that R is an **equivalence relation** on A if and only if R is reflexive, transitive, and symmetric.

2. **Remark.** To prove that a relation R is an equivalence relation on a set A, we prove that R is reflexive, symmetric, and transitive. Recall the following from the last lecture notes.

   To prove that R is reflexive: Suppose that $x \in A$. Prove that xRx.

   To prove that R is symmetric: Suppose that $x, y \in A$, and xRy. Prove that yRx.

   To prove that R is transitive: Suppose that $x, y, z \in A$. Suppose also that xRy and yRz. Then prove that xRz.

3. **Problem.** Define a relation R on $\mathbb{Z}$ by

   $$xRy \text{ if and only if } x^2 + y^2 \text{ is even.}$$

   Prove that R is an equivalence relation.

   Proof.

   First, we prove that R is reflexive. Suppose that $x \in \mathbb{Z}$. Observe that $x^2 + x^2 = 2x^2$. It follows that $x^2 + x^2$ is even. Thus, xRx. Therefore, R is reflexive.

   Second, we prove that R is symmetric. Suppose that $x, y \in \mathbb{Z}$, and xRy. Then $x^2 + y^2$ is even. Since $y^2 + x^2 = x^2 + y^2$, it follows that $y^2 + x^2$ is even. Thus, yRx. Therefore, R is symmetric.

   Third, we prove that R is transitive. Suppose that $x, y, z \in \mathbb{Z}$. Suppose also that xRy and yRz. Then $x^2 + y^2$ is even, and $y^2 + z^2$ is even.

Now, it follows from the definition, that the integer $-2y^2$ is even. Since the sum of three even integers is even, it follows that the integer,

$$x^2 + y^2 + y^2 + z^2 - 2y^2$$

is even. So $x^2 + z^2$ is even. Thus, xRz. Therefore, R is transitive.

□

4. **Definition.** Suppose that R is an equivalence relation on A. Suppose that $x \in A$. The **equivalence class** of x denoted $[x]$ is given by

$$[x] = \{y \in A : xRy\}.$$

Let's consider the equivalence relation R on $\mathbb{Z}$ given in the problem above. Consider the equivalence class of the integer 0. We have

$$[0] = \{y \in \mathbb{Z} : 0Ry\} = \{y \in \mathbb{Z} : 0^2 + y^2 \text{ is even}\}$$
$$= \{y \in \mathbb{Z} : y^2 \text{ is even}\} = \{y \in \mathbb{Z} : y \text{ is even}\}.$$

So the equivalence class of 0 is the set of all even integers.

Suppose we consider an even integer $y \neq 0$. What is the equivalence class of y? We know that y is in the equivalence class of 0, so we have $0Ry$.

I claim that $[0] \subseteq [y]$. Let's prove this claim. Suppose that $x \in [0]$. Then $0Rx$. Since R is symmetric we also have $xR0$. Now, we saw above that $0Ry$. Since $xR0$ and $0Ry$, we also have xRy (using the fact that R is transitive). Since R is symmetric we also have yRx. So, by definition, we have $x \in [y]$. Therefore, $[0] \subseteq [y]$.

I claim that also $[y] \subseteq [0]$. This can be proved in the same way as the previous claim. It follows that $[y] = [0]$. So, we have proved that for any even integer y, the equivalence class of y is the set of even integers. So, if E is the set of even integers, then

$$E = [0] = [2] = [-2] = [4] = [-4] = \ldots$$

Now, let's think about the equivalence class of the integer 1. We have

$$[1] = \{y \in \mathbb{Z} : 1Ry\} = \{y \in \mathbb{Z} : 1^2 + y^2 \text{ is even }\}$$

$$= \{y \in \mathbb{Z} : 1 + y^2 \text{ is even }\} = \{y \in \mathbb{Z} : y \text{ is odd }\}.$$

So, the equivalence class of 1 is the set of all odd integers. Moreover, we can show as above, that for any odd integer y, we have $[y] = [1]$. So if D denotes the set of odd integers, then

$$D = [1] = [-1] = [3] = [-3] = [5] = [-5] = \dots$$

□

By almost the same proof as above, we can prove the following result about equivalence relation and equivalence classes.

5. **Theorem.** Suppose that R is an equivalence relation on A. Suppose that $v, w \in A$. If $w \in [v]$, then $[w] = [v]$.

   Try to write a proof of this Theorem, proving that each of the two sets $[v]$ and $[w]$ is a subset of the other. I will include a proof in the lecture notes next time.