Sets and Logic, Dr. Block, Lecture Notes, 4-8-2020

I will discuss Problem 1 on Problem set 2.

1. Suppose that A, B, and C are sets. Prove that

$$(A \cup C) \subseteq (B \cup C)$$

if and only if

$$(A - C) \subseteq (B - C).$$

First, we have a hypothesis given, in the first sentence: Suppose that A, B, and C are sets. We could write this down as the first line of the proof. I would take this as understood, if you do not write this down.

Then, we note that we are asked to prove a statement of the form $P \Leftrightarrow Q$. Recall that the most common way to do this is to prove that both $P \Rightarrow Q$ and $Q \Rightarrow P$. It is best to put each of these statements with the proof in a separate paragraph. We sometimes call the two statements the two directions of the proof. Each direction may be proved by direct proof, contrapositive proof, or proof by contradiction. We do not have to use the same method for each direction.

Now, let's look at the first direction. Here is the statement that we need to prove.

If $(A \cup C) \subseteq (B \cup C)$, then $(A - C) \subseteq (B - C)$. This is a statement of the form $P \Rightarrow Q$. The "if" part is the P. This is the **hypothesis** of this statement. So we write this down in our proof as follows:

Suppose that $(A \cup C) \subseteq (B \cup C)$.

Now we are at the key point. Where do we go from here? To decide this:

We think about what we want to prove!

We want to prove that $(A - C) \subseteq (B - C)$. We ask ourself the question: How do we prove this? We recall that the logical form of this statement is:

$$\forall x, [(x \in (A - C)) \Rightarrow (x \in (B - C))]$$

This is again an "if-then" statement: If $x \in (A-C)$, then $x \in (B-C)$. This gives us an additional hypothesis. So the next line of the proof should be:

Suppose that $x \in (A - C)$.

Now, let's write down what our proof looks like up to this point.

Proof. Suppose that A, B, and C are sets.

First, we prove that if $(A \cup C) \subseteq (B \cup C)$, then $(A - C) \subseteq (B - C)$.

Suppose that $(A \cup C) \subseteq (B \cup C)$. Suppose that $x \in (A - C)$.

□

Now again, we think about what we want to prove, namely $x \in (B-C)$. At this stage, we have to figure out how to get what we want from what we have so far. Now, let's think about how we might use the hypothesis:

$$(A \cup C) \subseteq (B \cup C).$$

What does this tell us: If we know that for some y it is the case that $y \in (A \cup C)$, then it will follow that $y \in (B \cup C)$. This hypothesis does not give us any y. We need to find a suitable y. Looking at our proof so far, suggest the following: Use the x for y. With this understanding it is not difficult to see the proof. Of course we have to know all of the relevant definitions: $\cup$, $\subseteq$ and so on. Here is the entire proof of this direction.

Proof. Suppose that A, B, and C are sets.

First, we prove that if $(A \cup C) \subseteq (B \cup C)$, then $(A - C) \subseteq (B - C)$.

Suppose that $(A \cup C) \subseteq (B \cup C)$. Suppose that $x \in (A-C)$. Then $x \in A$ and $x \notin C$. Since $x \in A$, we have $x \in (A \cup C)$. Since $(A \cup C) \subseteq (B \cup C)$, it follows that $x \in (B \cup C)$. Finally, as $x \notin C$ and $x \in (B \cup C)$, it follows that $x \in (B - C)$. Therefore, $(A - C) \subseteq (B - C)$.

□

Now, let's look at the second direction of the proof:

If $(A-C) \subseteq (B-C)$, then $(A \cup C) \subseteq (B \cup C)$. Again, we will first write down the hypothesis of the statement: $(A-C) \subseteq (B-C)$. Then, looking at what we want to prove, we will write down: Suppose $x \in (A \cup C)$.

So we have at this stage the following:

Proof of second direction.

Second, we prove that if $(A - C) \subseteq (B - C)$, then $(A \cup C) \subseteq (B \cup C)$. Suppose that $(A - C) \subseteq (B - C)$. Suppose that $x \in (A \cup C)$.

$\Box$

Now, from the hypothesis, $(A - C) \subseteq (B - C)$, we see that if we have some $y \in (A - C)$, then we can say that $y \in (B - C)$. So, let's think about this: Could we use our x that we have above for the y. We know that $x \in (A \cup C)$. So either $x \in A$ or $x \in C$. Knowing just this, we can not conclude that $x \in (A - C)$.

At this stage, we should again think about what we want to prove. We want to prove that $x \in (B \cup C)$. So we want to prove that either $x \in B$ or $x \in C$. How do we prove something of this form? There are different ways we can do this.

Recall the ways to prove a statement of the form $P \vee Q$.

* Suppose $\sim P$ and prove Q.

* Suppose $\sim Q$ and prove P.

* Proceed by contradiction.

* Make cases and deal with each case separately. When you do this the cases must cover every possibility.

Let's try to make cases. We might try 2 cases as follows:

Case 1. $x \in A$.

Case 2. $x \in C$.

These two cases do cover all of the possilibies as we have $x \in (A \cup C)$. Could we prove that in each of these cases $x \in (B \cup C)$? We observe that in Case 2, it would follow from the definition of union that $x \in (B \cup C)$. What about Case 1? Knowing that $x \in A$ and using our hypothesis that $(A - C) \subseteq (B - C)$, can we deduce that $x \in (B \cup C)$? The problem is that we can not say that $x \in (A - C)$. We can solve this problem by changing Case 1 (and leaving Case 2 as is). So we will have 2 cases as follows:

Case 1. $x \notin C$.

Case 2. $x \in C$.

So now we can write the proof.

Proof of second direction.

Second, we prove that if $(A-C) \subseteq (B-C)$, then $(A\cup C) \subseteq (B\cup C)$. Suppose that $(A-C) \subseteq (B-C)$. Suppose that $x \in (A\cup C)$. We consider two cases.

Case 1. $x \notin C$. Then $x \in (A-C)$. Since $(A-C) \subseteq (B-C)$, it follows that $x \in (B-C)$. So $x \in B$, and thus, $x \in (B \cup C)$.

Case 2. $x \in C$. Then $x \in (B \cup C)$.

So, in all cases, we have $x \in (B \cup C)$. Therefore, $(A \cup C) \subseteq (B \cup C)$.

□

Here is the entire proof with both cases included.

Proof. Suppose that A, B, and C are sets.

First, we prove that if $(A \cup C) \subseteq (B \cup C)$, then $(A - C) \subseteq (B - C)$.

Suppose that $(A\cup C) \subseteq (B\cup C)$. Suppose that $x \in (A-C)$. Then $x \in A$ and $x \notin C$. Since $x \in A$, we have $x \in (A \cup C)$. Since $(A \cup C) \subseteq (B \cup C)$, it follows that $x \in (B \cup C)$. Finally, as $x \notin C$ and $x \in (B \cup C)$, it follows that $x \in (B - C)$. Therefore, $(A - C) \subseteq (B - C)$.

Second, we prove that if $(A-C) \subseteq (B-C)$, then $(A\cup C) \subseteq (B\cup C)$. Suppose that $(A-C) \subseteq (B-C)$. Suppose that $x \in (A\cup C)$. We consider two cases.

Case 1. $x \notin C$. Then $x \in (A-C)$. Since $(A-C) \subseteq (B-C)$, it follows that $x \in (B-C)$. So $x \in B$, and thus, $x \in (B \cup C)$.

Case 2. $x \in C$. Then $x \in (B \cup C)$.

So, in all cases, we have $x \in (B \cup C)$. Therefore, $(A \cup C) \subseteq (B \cup C)$.

□

Here is another proof of the second direction. In this proof, instead of proving directly that $x \in (B \cup C)$, we prove the equivalent statement: If $x \notin C$, then $x \in B$. This comes from one of the ways to prove a statement of the form $P \vee Q$ above.

Proof of second direction (Alternate 1).

Second, we prove that if $(A - C) \subseteq (B - C)$, then $(A \cup C) \subseteq (B \cup C)$. Suppose that $(A - C) \subseteq (B - C)$. Suppose that $x \in (A \cup C)$.

Suppose that $x \notin C$. Then $x \in (A - C)$. Since $(A - C) \subseteq (B - C)$, it follows that $x \in (B - C)$. So $x \in B$.

We conclude that if $x \notin C$, then $x \in B$. Thus, $x \in (B \cup C)$. Therefore, $(A \cup C) \subseteq (B \cup C)$.

□

Another way to attack the second direction is to prove the contrapositive. Recall the following: The logical form of $(A \cup C) \subseteq (B \cup C)$ is:

$$\forall x, [(x \in (A \cup C)) \Rightarrow (x \in (B \cup C))]$$

The logical form of the negation is:

$$\exists x, [(x \in (A \cup C)) \wedge (x \notin (B \cup C))]$$

So, in our proof, we will start with some x with

$$[(x \in (A \cup C)) \wedge (x \notin (B \cup C))].$$

We will want to find some y with

$$[(y \in (A - C)) \wedge (y \notin (B - C))].$$

Taking x for the y will work.

Here is the proof.

Proof of second direction (Alternate 2).

Second, we prove that if $(A - C) \subseteq (B - C)$, then $(A \cup C) \subseteq (B \cup C)$. We prove the contrapositive. Suppose that $(A \cup C) \nsubseteq (B \cup C)$. Then there is some $x \in (A \cup C)$ such that $x \notin (B \cup C)$. Since $x \notin (B \cup C)$, it follows that $x \notin C$. Since $x \in (A \cup C)$ and $x \notin C$, we have $x \in (A - C)$.

Also, as $x \notin (B \cup C)$, we have $x \notin B$. Thus, $x \notin (B - C)$. Since we also have $x \in (A - C)$, it follows that $(A - C) \nsubseteq (B - C)$.

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