1. (5 points) Construct a truth table for the formula \((P \land \sim Q) \Rightarrow R\).

\[
\begin{array}{cccccccc}
P & Q & R & \sim Q & P \land \sim Q & (P \land \sim Q) \Rightarrow R \\
T & T & T & F & F & T \\
T & T & F & F & F & T \\
T & F & T & T & T & T \\
T & F & F & T & F & F \\
F & T & T & F & F & T \\
F & T & F & F & F & T \\
F & F & T & T & F & T \\
F & F & F & T & F & T \\
\end{array}
\]

2. (3 points) Negate the following statement.

For every integer \(x\), if 9 divides \(x^2 - 7x\), then 3 divides \(x\).

**answer:** There exists an integer \(x\) such that 9 divides \(x^2 - 7x\), and 3 does not divide \(x\).

3. (3 points) Determine if the statement is true or false.

\[
\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + 2y = 5
\]

**answer:** False

4. (3 points) Determine if the statement is true or false. The logical formulas \(P \lor Q\) and \(\sim P \Rightarrow Q\) are equivalent.

**answer:** True

5. (3 points) Determine if the statement is true or false. For any set \(A\), we have \(A \subseteq \mathcal{P}(A)\).

**answer:** False
6. (3 points) Determine if the statement is true or false. Suppose that \(R_1\) and \(R_2\) are relations on \(A\). If \(R_1\) and \(R_2\) are transitive, then \(R_1 \cap R_2 \) is transitive.

**answer:** True

7. (3 points) Determine if the statement is true or false. Let \(A = \mathcal{P}(\mathbb{R})\), and define \(f : \mathbb{R} \rightarrow A\) by the formula \(f(x) = \{y \in \mathbb{R} : y^2 < x\}\). Then \(f\) is injective.

**answer:** False

8. (6 points) Prove that if \(A, B, \) and \(C\) are sets, then

\[
A - (B \cup C) = (A - B) \cap (A - C).
\]

**Proof:** The following are equivalent:

\[
x \in A - (B \cup C)
\]

\[
x \in A \land \lnot (x \in B \lor x \in C)
\]

\[
x \in A \land x \notin B \land x \notin C
\]

\[
x \in (A - B) \cap (A - C).
\]

It follows that

\[
A - (B \cup C) = (A - B) \cap (A - C).
\]

9. (6 points) Prove that for any sets \(A\) and \(B\), if \(\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)\) then either \(A \subseteq B\) or \(B \subseteq A\).

**Proof:**

We prove the contrapositive. Suppose that \(A\) is not a subset of \(B\) and \(B\) is not a subset of \(A\). There exists \(x \in A - B\) and \(y \in B - A\). Set \(D = \{x, y\}\). Then \(D \in \mathcal{P}(A \cup B)\) but \(D \notin (\mathcal{P}(A) \cup \mathcal{P}(B))\). It follows that \(\mathcal{P}(A \cup B)\) is not a subset of \((\mathcal{P}(A) \cup \mathcal{P}(B))\).

10. (6 points)
Suppose that $X$ is a nonempty set. Let $A$ denote the set of functions from $X$ to $X$. Let $B$ denote the set of $h \in A$ such that $h$ is bijective. Define a relation $R$ on $A$ by

$$R = \{(f, g) \in A \times A : \exists h \in B \text{ such that } h \circ f = g \circ h\}.$$ 

Prove that $R$ is an equivalence relation on $A$.

**proof:**

First, we prove that $R$ is reflexive. Suppose that $f \in A$. The identity function $i_X : X \to X$ is bijective, so $i_X \in B$. Moreover, $i_X \circ f = f \circ i_X$. It follows that $(f, f) \in R$. Since $f$ was arbitrary, $R$ is reflexive.

Second, we prove that $R$ is symmetric. Suppose that $f, g \in A$, and $(f, g) \in R$. There is some $h \in B$ with $h \circ f = g \circ h$. Since $h$ is bijective, $h^{-1}$ exists and $h^{-1} \in B$. We have

$$f = h^{-1} \circ h \circ f = h^{-1} \circ g \circ h.$$

It follows that

$$f \circ h^{-1} = h^{-1} \circ g \circ h \circ h^{-1} = h^{-1} \circ g.$$

Thus, $(g, f) \in R$. Since $f$ and $g$ were arbitrary, $R$ is symmetric.

Third, we prove that $R$ is transitive. Suppose that $f, g, k \in A$, $(f, g) \in R$, and $(g, k) \in R$. For some $h_1, h_2 \in B$ we have

$$h_1 \circ f = g \circ h_1, \text{ and } h_2 \circ g = k \circ h_2.$$

Set $h_3 = h_2 \circ h_1$. Then $h_3 \in B$, and

$$h_3 \circ f = h_2 \circ h_1 \circ f = h_2 \circ g \circ h_1 = k \circ h_2 \circ h_1 = k \circ h_3.$$

It follows that $(f, k) \in R$. Since $f, g$, and $k$ were arbitrary, $R$ is transitive.

Finally, since $R$ is reflexive, transitive, and symmetric, $R$ is an equivalence relation.

11. (6 points) Let $A = \mathbb{R} - \{2\}$, and let $B = \mathbb{R} - \{3\}$. Let $f : A \to B$ be defined by the formula $f(x) = \frac{3x}{x-2}$. Prove that $f$ is bijective. Also, find a formula for $f^{-1}$. 
Proof: First, we verify that for all $x \in A$ it is the case that $f(x) \in B$. We prove this by contradiction. Suppose that for some $x \in A$, we have $f(x) \notin B$. Then $3 = \frac{3x}{x-2}$. It follows that $3x - 6 = 3x$, a contradiction.

We claim that for all $y \in B$ we have $2y - 3 \neq 2$. We prove this claim by contradiction. Suppose that for some $y \in B$ we have $2y - 3 = 2$. Then $2y = 6$, a contradiction. This proves the claim. It follows from the claim that we have a function $g : B \to A$ given by the formula $g(y) = \frac{2y}{y-3}$.

Next, we prove that $g \circ f = i_A$. We have that both $g \circ f$ and $i_A$ are functions from $A$ to $A$. Suppose that $x \in A$. Then

$$(g \circ f)(x) = g(f(x)) = g(\frac{3x}{x-2}) = \frac{2 \cdot \frac{3x}{x-2}}{3x - 3x + 6} = x.$$ 

It follows that $g \circ f = i_A$.

Finally, we prove that $f \circ g = i_B$. We have that both are functions from $B$ to $B$. Suppose that $y \in B$. Then

$$(f \circ g)(y) = f(g(y)) = f(\frac{2y}{y-3}) = \frac{3 \cdot \frac{2y}{y-3}}{2y - 2y + 6} = y.$$ 

It follows that $f \circ g = i_B$.

Now, we have $f : A \to B$, $g : B \to A$, $g \circ f = i_A$, and $f \circ g = i_B$. It follows from a previous theorem that $f$ is bijective and $g = f^{-1}$. So a formula for $f^{-1}$ is given by $f^{-1}(y) = \frac{2y}{y-3}$.

12. (6 points) Prove the following by mathematical induction: For all $n \in \mathbb{N}$,

$$9 \mid (4^n + 6n - 1).$$

Recall that this notation means that 9 divides $(4^n + 6n - 1)$.

Proof: For $n \in \mathbb{N}$ we let $P(n)$ be the statement $9 \mid (4^n + 6n - 1)$. We use induction to prove that $P(n)$ is true for all $n \in \mathbb{N}$. First, $P(1)$ says that 9 divides $4^1 + 6 \cdot 1 - 1$, which is true. Now let $n \in \mathbb{N}$ and assume that $P(n)$ is true. Then for some $q \in \mathbb{Z}$ we have $4^n + 6n - 1 = 9q$. We need to prove that $P(n+1)$ is true, that is, $9 \mid (4^{n+1} + 6(n+1) - 1)$. We have

$$4^{n+1} + 6(n+1) - 1 = 4 \cdot (4^n + 6n - 1) + 13.$$ 

Since $4^n + 6n - 1 = 9q$, we have $4 \cdot 9q + 13 = 36q + 13 = 9(4q + 1) + 4$. It follows that $9 \mid (4^{n+1} + 6(n+1) - 1)$, and thus $P(n+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. 


9q. Hence $4^n = 9q - 6n + 1$, so $4^{n+1} = 4 \cdot 4^n = 36q - 24n + 4$. It follows that

$$4^{n+1} + 6(n+1) - 1 = 36q - 24n + 4 + 6n + 6 - 1 = 36q - 18n + 9 = 9(4q - 2n + 1).$$

Since $4q - 2n + 1 \in \mathbb{Z}$ we see that $9 \mid (4^{n+1} + 6(n + 1) - 1)$. Therefore $P(n + 1)$ is true. It follows by induction that $P(n)$ is true for all $n \in \mathbb{N}$. 