Advanced Calculus I, Dr. Block, Chapter 3 notes

1. Definition. Suppose D is a subset of $\mathbb{R}$ which is not bounded above. Let $f : D \to \mathbb{R}$, and let $L \in \mathbb{R}$. We say that $\lim_{x\to\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in D$ with $x \geq M$.

2. Theorem. Suppose D is a subset of $\mathbb{R}$ which is not bounded above. Let $f : D \to \mathbb{R}$, and let $L \in \mathbb{R}$. Then $\lim_{x\to\infty} f(x) = L$ if and only if for every sequence $\{x_n\}$ of points in D such that $\lim_{n\to\infty} x_n = \infty$ we have $\lim_{n\to\infty} f(x_n) = L$.

3. Example. Show that $lim_{x\to\infty} \sin x$ does not exist.

4. Example. Show that $lim_{x\to\infty} \frac{1}{x} = 0$.

5. Definition. Suppose D is a subset of $\mathbb{R}$ which is not bounded below. Let $f : D \to \mathbb{R}$, and let $L \in \mathbb{R}$. We say that $\lim_{x\to-\infty} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $M < 0$ such that $|f(x) - L| < \epsilon$ for all $x \in D$ with $x \leq M$.

6. Theorem. (sums, product quotients, roots, etc.) See Theorem 3.1.7 on page 120 of the text.

7. Definition. Suppose D is a subset of $\mathbb{R}$ which is not bounded above. Let $f : D \to \mathbb{R}$. We say that $\lim_{x\to\infty} f(x) = \infty$ if and only if for every $B > 0$ there exists $M > 0$ such that $f(x) > B$ for all $x \in D$ with $x \geq M$.

8. Definition. Suppose D is a subset of $\mathbb{R}$ which is not bounded above. Let $f : D \to \mathbb{R}$. We say that $\lim_{x\to\infty} f(x) = -\infty$ if and only if for every $B < 0$ there exists $M > 0$ such that $f(x) < B$ for all $x \in D$ with $x \geq M$.

9. Definition. Suppose D is a subset of $\mathbb{R}$ which is not bounded below. Let $f : D \to \mathbb{R}$, and let $L \in \mathbb{R}$. We say that $\lim_{x\to-\infty} f(x) = \infty$ if and only if for every $B > 0$ there exists $M < 0$ such that $f(x) > B$ for all $x \in D$ with $x \leq M$.

10. Definition. Suppose D is a subset of $\mathbb{R}$ which is not bounded below. Let $f : D \to \mathbb{R}$, and let $L \in \mathbb{R}$. We say that $\lim_{x\to-\infty} f(x) = -\infty$ if and only if for every $B < 0$ there exists $M < 0$ such that $f(x) < B$ for all $x \in D$ with $x \leq M$.

11. Squeeze Theorem (different from statement in text). Suppose D is a subset of $\mathbb{R}$ which is not bounded above. Let $f, g, h : D \to \mathbb{R}$. Suppose that there exists $M > 0$ such that

$$f(x) \leq g(x) \leq h(x).$$

for all $x \in D$ with $x \geq M$. Suppose also that

$$lim_{x\to\infty} f(x) = A = lim_{x\to\infty} h(x).$$

Then $lim_{x\to\infty} g(x) = A$.

12. Theorem. Suppose that $a > 0$.
If $a < 1$, then $lim_{x\to\infty} a^x = 0$.
If $a = 1$, then $lim_{x\to\infty} a^x = 1$.
If $a > 1$, then $lim_{x\to\infty} a^x = \infty$.

13. Definition. Suppose D is a subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$. Let $a \in \mathbb{R}$, and suppose that a is an accumulation point of D. Let $L \in \mathbb{R}$. We say that $\lim_{x\to a} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in D$ with $x \in N_\delta^-(a)$.

14. Definition. Suppose D is a subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$. Let $a \in \mathbb{R}$, and suppose that a is an accumulation point of D. We say that $\lim_{x\to a} f(x) = \infty$ if and only if for every $B > 0$ there exists $\delta > 0$ such that $f(x) > B$ for all $x \in D$ with $x \in N_\delta^-(a)$.

15. Definition. Suppose D is a subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$. Let $a \in \mathbb{R}$, and suppose that a is an accumulation point of D. We say that $\lim_{x\to a} f(x) = -\infty$ if and only if for every $B < 0$ there exists $\delta > 0$ such that $f(x) < B$ for all $x \in D$ with $x \in N_\delta^-(a)$.

16. Theorem (substitution). Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to b} g(x) = a$. Suppose that either $f(a) = L$ or $g(x) \neq a$ for all x in some deleted neighborhood of b. Then

$$\lim_{x\to b} f(g(x)) = L.$$

17. Definition. Suppose D is a subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$. Let $a \in \mathbb{R}$, and suppose that a is an accumulation point of $D \cap (a, \infty)$. Let $L \in \mathbb{R}$. We say that $\lim_{x\to a^+} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in D$ with $x \in (N_\delta^-(a) \cap (a, \infty))$.

18. Note. $N_\delta^-(a) \cap (a, \infty) = (a, a + \delta)$.

19. Definition. Suppose D is a subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$. Let $a \in \mathbb{R}$, and suppose that a is an accumulation point of $D \cap (a, \infty)$. We say that $\lim_{x\to a^+} f(x) = \infty$ if and only if for every $B > 0$ there exists $\delta > 0$ such that $f(x) > B$ for all $x \in D$ with $x \in (N_\delta^-(a) \cap (a, \infty))$.

20. Definition. Suppose D is a subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$. Let $a \in \mathbb{R}$, and suppose that a is an accumulation point of $D \cap (a, \infty)$. We say that $\lim_{x\to a^+} f(x) = -\infty$ if and only if for every $B < 0$ there exists $\delta > 0$ such that $f(x) < B$ for all $x \in D$ with $x \in (N_\delta^-(a) \cap (a, \infty))$.