Advanced Calculus I, Dr. Block, Chapter 4 notes

1. Definition. Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$. Let $a \in D$. We say that f is continuous at a if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for all $x \in D$ with $|x - a| < \delta$.

2. Theorem. Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$. Let $a \in D$. Then f is continuous at a if and only if either one of the following hold:

(i). a is not an accumulation point of D.

(ii). a is an accumulation point of D, and $\lim_{x \to a} f(x) = f(a)$.

3. Definition. Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$. We say that f is continuous if and only if f is continuous at a for every $a \in D$.

4. Theorem. Let D be a subset of $\mathbb{R}$, and let $f, g : D \to \mathbb{R}$. Let $a \in D$. If f and g are continuous at a then so are $f + g$, $f - g$, $f \cdot g$. Also, if $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a.

5. Theorem. The following functions are continuous: polynomials, rational functions, sine function, cosine function, exponenial function, square root function, natural logarithm function, and arctan function.

6. Theorem. If f is continuous at a and g is continuous at f(a), then $g \circ f$ is continuous at a.

7. Theorem. Suppose that f is continuous at a. Let $\{t_n\}$ be a sequence of points in the domain of f such that $\lim_{n \to \infty} t_n = a$. Then $\lim_{n \to \infty} f(t_n) = f(a)$.

8. Definition. Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$. Let a be an accumulation point of D. We say that a is a point of discontinuity of f if either one of the following hold:

(i). $a \notin D$.

(ii). $a \in D$, but f is not continuous at a.

9. Definition. Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$. Let a be a point of discontinuity of f. We say that a is a point of removable discontinuity if either one of the following hold:

(i). $a \notin D$, and $\lim_{x \to a} f(x) = L$ for some $L \in \mathbb{R}$.

(ii). $a \in D$, and $\lim_{x \to a} f(x) = L$ for some $L \in \mathbb{R}$, but $L \neq f(a)$.

10. Definition. Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$. Let a be a point of discontinuity of f. We say that f has a jump discontinuity at $x = a$ if and only if there exists real numbers M and E with $M \neq E$ such that

$$\lim_{x \to a^-} f(x) = M, \; \lim_{x \to a^+} f(x) = E.$$

11. Definition. Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$. Let a be a point of discontinuity of f. We say that f has an infinite discontinuity at $x = a$ if and only if one of the following holds:

(i). $\lim_{x\to a^+} f(x) = \pm\infty$ and $\lim_{x\to a^-} f(x) = \pm\infty$.

(ii). a is not an accumulation point of $D \cap (a, \infty)$ and $\lim_{x\to a^-} f(x) = \pm\infty$.

(ii). a is not an accumulation point of $D \cap (-\infty, a)$ and $\lim_{x\to a^+} f(x) = \pm\infty$.

12. Definition. Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$. Let a be a point of discontinuity of f. If none of the definitions 9, 10, 11 apply to a, we call the discontinuity oscillating.

13. Definition. Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$. We say that f is bounded on D if and only if there is a real number B such that $|f(x)| \leq B$ for all $x \in D$.

14. Theorem. If a function f is continuous on a closed bounded interval $[a, b]$, then f is bounded on $[a, b]$.

15. Definition. Let D be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$. We say that f attains its maximum value on D if and only if there exists $v \in D$ such that $f(v) \geq f(x)$ for all $x \in D$. We say that f attains its minimum value on D if and only if there exists $w \in D$ such that $f(w) \leq f(x)$ for all $x \in D$.

16. Theorem. (Extreme Value Theorem) If a function f is continuous on a closed bounded interval $[a, b]$, then f attains its maximum and minimum values on $[a, b]$.

17. Theorem. (Intermediate Value Theorem) If a function f is continuous on a closed bounded interval $[a, b]$ and k is any real number between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ with $f(c) = k$.
Here the hypothesis "k is between $f(a)$ and $f(b)$" means that
either $f(a) < k < f(b)$ or $f(a) > k > f(b)$.

18. Corollary. Suppose that a and b are real numbers with $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous and not constant, then the range of f is an interval $[c, d]$, where c and d are real numbers with $c < d$.