Advanced Calculus I, Dr. Block, Chapter 2 notes

1. Theorem. (Archimedean Property) Let \( x \) be any real number. There exists a positive integer \( n^* \) greater than \( x \).

2. Definition. A sequence is a real-valued function whose domain consists of all integers which are greater than or equal to some fixed integer (which is often 1). The notation \( \{a_n\} \) is used.

3. Definition. We say that a sequence \( \{a_n\} \) converges to a real number \( L \) if and only if for every \( \epsilon > 0 \), there exists a positive integer \( n^* \) such that for all \( n \geq n^* \) we have \( |a_n - L| < \epsilon \). The real number \( L \) is called the limit of the sequence and we write \( \lim_{n \to \infty} a_n = L \).

   We also say that the sequence is convergent.

   If there is no real number \( L \) as above, we say that the sequence diverges or is divergent.

4. Problem. Prove using the definition that \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

   Formal Proof. Let \( \epsilon > 0 \). By the Archimedean Property there exists a positive integer \( n^* > \frac{1}{\epsilon} \). If \( n \geq n^* \) we have
   \[
   \left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{n^*} < \epsilon.
   \]

5. Problem. Prove using the definition that \( \lim_{n \to \infty} \frac{5n}{n^2 + 1} = 0 \).

   Preliminary consideration: We want \( \left| \frac{5n}{n^2 + 1} - 0 \right| < \epsilon \) for \( n \geq n^* \). We see that
   \[
   \left| \frac{5n}{n^2 + 1} - 0 \right| = \frac{5n}{n^2 + 1} \leq \frac{5n}{n^2} = \frac{5}{n}.
   \]

   Also we will have \( \frac{5}{n} < \epsilon \) if \( n > \frac{5}{\epsilon} \).

   Formal Proof. Let \( \epsilon > 0 \). By the Archimedean Property there exists a positive integer \( n^* > \frac{5}{\epsilon} \). If \( n \geq n^* \) we have
   \[
   \left| \frac{5n}{n^2 + 1} - 0 \right| = \frac{5n}{n^2 + 1} \leq \frac{5n}{n^2} = \frac{5}{n} \leq \frac{5}{n^*} < \epsilon.
   \]

6. Note. A sequence \( \{a_n\} \) diverges if and only if for every real number \( L \) there exists \( \epsilon > 0 \) such that for every positive integer \( n^* \) there exists \( n \geq n^* \) with \( |a_n - L| \geq \epsilon \).

7. Theorem. Any two limits of a convergent sequence are the same. (If a sequence converges, then the limit of the sequence is unique.)

8. Definition. We say that a sequence \( \{a_n\} \) is bounded if and only if there is a real number \( B \) such that \( |a_n| \leq B \) for all \( n \).
9. Theorem. Any convergent sequence is bounded.

10. Theorem. If $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$ with $A, B \in \mathbb{R}$, then
    1. $\lim_{n \to \infty} a_n + b_n = A + B$.
    2. $\lim_{n \to \infty} a_n - b_n = A - B$.
    3. $\lim_{n \to \infty} a_n \cdot b_n = A \cdot B$.
    4. $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{A}{B}$, if $B \neq 0$.
    5. $\lim_{n \to \infty} (a_n)^p = A^p$, for any positive rational number $p$, provided that the "roots" are defined.

11. Theorem. (Squeeze Theorem) Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences, and suppose that there exists a positive integer $K$ such that if $n \geq K$, then $a_n \leq b_n \leq c_n$. Suppose that for some real number $L$

$$
\lim_{n \to \infty} a_n = L = \lim_{n \to \infty} c_n.
$$

Then $\lim_{n \to \infty} b_n = L$.

12. Theorem. If a sequence $\{a_n\}$ converges to 0 and a sequence $\{b_n\}$ is bounded, then the sequence $\{a_n \cdot b_n\}$ converges to 0.

13. Theorem. (Special limits to remember and use.)
    1. If $p > 0$, then $\lim_{n \to \infty} \frac{1}{n^p} = 0$.
    2. If $|r| < 1$, then $\lim_{n \to \infty} r^n = 0$.
    3. If $c > 0$, then $\lim_{n \to \infty} \sqrt[n]{c} = 1$.
    4. $\lim_{n \to \infty} \sqrt[n]{n} = 1$.
    5. If $\lim_{n \to \infty} a_n = 0$, then $\lim_{n \to \infty} \sin(a_n) = 0$.
    6. If $\lim_{n \to \infty} a_n = 0$, then $\lim_{n \to \infty} \frac{\sin(a_n)}{a_n} = 1$.

14. Definition. We say the sequence $\{a_n\}$ diverges to $\infty$ if and only if for every $M > 0$, there is a positive integer $n^*$ such that for all $n \geq n^*$ we have $a_n > M$. In this case we write

$$
\lim_{n \to \infty} a_n = \infty.
$$

15. Definition. We say the sequence $\{a_n\}$ diverges to $-\infty$ if and only if for every $M < 0$, there is a positive integer $n^*$ such that for all $n \geq n^*$ we have $a_n < M$. In this case we write

$$
\lim_{n \to \infty} a_n = -\infty.
$$
16. Theorem. If \( \lim_{n \to \infty} a_n = \infty \) and there exists a positive integer \( K \) such that \( b_n \geq a_n \) for all \( n \geq k \), then \( \lim_{n \to \infty} b_n = \infty \).

17. Theorem. If \( \lim_{n \to \infty} a_n = -\infty \) and there exists a positive integer \( K \) such that \( b_n \leq a_n \) for all \( n \geq k \), then \( \lim_{n \to \infty} b_n = -\infty \).

18. Theorem. Suppose that \( \lim_{n \to \infty} a_n = \infty \).
1. If \( \{b_n\} \) is bounded below, then \( \lim_{n \to \infty} (a_n + b_n) = \infty \).
2. If \( \{b_n\} \) converges or diverges to \( \infty \), then \( \lim_{n \to \infty} (a_n + b_n) = \infty \).
3. If \( \{b_n\} \) is bounded below by a positive number, then \( \lim_{n \to \infty} (a_n \cdot b_n) = \infty \).
4. If \( \{b_n\} \) converges to a positive number or diverges to \( \infty \), then
   \[
   \lim_{n \to \infty} (a_n \cdot b_n) = \infty.
   \]
5. If \( \{b_n\} \) converges to a negative number or diverges to \( -\infty \), then
   \[
   \lim_{n \to \infty} (a_n \cdot b_n) = -\infty.
   \]

19. Theorem.
1. If \( \lim_{n \to \infty} a_n = \infty \), then \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \).
2. If \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \) and \( a_n > 0 \) for all \( n \) sufficiently large, then
   \[
   \lim_{n \to \infty} a_n = \infty.
   \]
3. If \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \) and \( a_n < 0 \) for all \( n \) sufficiently large, then
   \[
   \lim_{n \to \infty} a_n = -\infty.
   \]

20. Theorem. (Ratio Test) Suppose that \( \{a_n\} \) is a sequence of nonzero real numbers such that
   \[
   \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \alpha
   \]
where either \( \alpha \in \mathbb{R} \) or \( \alpha = \infty \).
1. If \( \alpha < 1 \), then \( \lim_{n \to \infty} a_n = 0 \).
2. If \( \alpha > 1 \), then \( \lim_{n \to \infty} |a_n| = \infty \), so the sequence \( \{a_n\} \) diverges.

21. Definition. We say that a sequence \( \{a_n\} \) oscillates if and only if none of the three statements below hold.
1. \( \lim_{n \to \infty} a_n = L \) for some \( L \in \mathbb{R} \).
2. \[ \lim_{n \to \infty} a_n = \infty. \]
3. \[ \lim_{n \to \infty} a_n = -\infty. \]

22. Definition. We say that a sequence \( \{a_n\} \) is increasing if and only if \( n < k \) implies \( a_n \leq a_k \).

23. Remark. A sequence \( \{a_n\} \) is increasing if and only if for all \( n \) we have \( a_n \leq a_{n+1} \).

24. Remark. A sequence \( \{a_n\} \) of positive real numbers is increasing if and only if for all \( n \) we have \( \frac{a_{n+1}}{a_n} \geq 1 \).

25. Definition. We say that a sequence \( \{a_n\} \) is eventually increasing if and only if there is a positive integer \( n^* \) such that \( n^* \leq n < k \) implies \( a_n \leq a_k \).

26. Definition. We say that a sequence \( \{a_n\} \) is decreasing if and only if \( n < k \) implies \( a_n \geq a_k \).

27. Remark. A sequence \( \{a_n\} \) is decreasing if and only if for all \( n \) we have \( a_n \geq a_{n+1} \).

28. Remark. A sequence \( \{a_n\} \) of positive real numbers is decreasing if and only if for all \( n \) we have \( \frac{a_{n+1}}{a_n} \leq 1 \).

29. Definition. We say that a sequence \( \{a_n\} \) is eventually decreasing if and only if there is a positive integer \( n^* \) such that \( n^* \leq n < k \) implies \( a_n \geq a_k \).

30. Theorem. A bounded, increasing sequence converges. An unbounded, increasing sequence diverges to \( \infty \).

31. Theorem. A bounded, decreasing sequence converges. An unbounded, decreasing sequence diverges to \( -\infty \).

32. Definition. We say that a sequence \( \{a_n\} \) is monotone if and only if either \( \{a_n\} \) is increasing or \( \{a_n\} \) is decreasing.

33. Definition. Let \( \epsilon > 0 \), and let \( s \in \mathbb{R} \). The \( \epsilon \)-neighborhood of \( s \) is
\[
N_\epsilon(s) = \{x \in \mathbb{R} : |x - s| < \epsilon \} = (s - \epsilon, s + \epsilon).
\]
The deleted \( \epsilon \)-neighborhood of \( s \) is
\[
N_\epsilon^-(s) = \{x \in \mathbb{R} : 0 < |x - s| < \epsilon \} = (s - \epsilon, s) \cup (s, s + \epsilon).
\]

34. Definition. Let \( S \subseteq \mathbb{R} \), and let \( w \in \mathbb{R} \). We say that \( w \) is an accumulation point of \( S \) if and only if every deleted neighborhood of \( w \) contains at least one point of \( S \).
35. Theorem. Let \( S \subseteq \mathbb{R} \), and let \( w \in \mathbb{R} \). Then \( w \) is an accumulation point of \( S \) if and only if every neighborhood of \( w \) contains infinitely many points of \( S \).

36. Theorem. (Bolzano-Weierstrass Theorem for sets) Every bounded infinite subset of \( \mathbb{R} \) has at least one accumulation point.

37. Definition. We say that a sequence \( \{a_n\} \) is a Cauchy sequence if and only if for every \( \epsilon > 0 \), there exists a positive integer \( n^\ast \) such that for all \( k, j \geq n^\ast \) we have \( |a_k - a_j| < \epsilon \).

38. Theorem. Let \( \{a_n\} \) be a sequence of real numbers. Then \( \{a_n\} \) is a Cauchy sequence if and only if \( \{a_n\} \) converges.

39. Definition. The sequence \( \{b_n\}_{n=i}^{\infty} \) is a subsequence of the sequence \( \{a_n\}_{n=j}^{\infty} \) if and only if there exists a strictly increasing function \( f : \{x \in \mathbb{N} : x \geq i\} \to \{x \in \mathbb{N} : x \geq j\} \) such that \( b_n = a_{f(n)} \) for all \( n \in \mathbb{N} \) with \( n \geq i \).

We sometimes use the notation \( b_k = a_{n_k} \) for a subsequence. In this case, \( n_k \) must be a strictly increasing function of \( k \).

40. Theorem. (Bolzano-Weierstrass Theorem for sequences) Every bounded sequence in \( \mathbb{R} \) has at least one convergent subsequence.

41. Definition. We let \( \mathbb{E} \) denote the set of extended real numbers defined by
\[
\mathbb{E} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}.
\]

42. Definition. Let \( \{a_n\} \) be a sequence of real numbers, and let \( A \in \mathbb{E} \). We say that \( A \) is a subsequential limit point of the sequence \( \{a_n\} \) if and only if there is a subsequence \( a_{n_k} \) of \( \{a_n\} \) such that
\[
\lim_{k \to \infty} a_{n_k} = A.
\]

43. Theorem. Let \( \{a_n\} \) be a sequence of real numbers. There exists a largest subsequential limit point of the sequence and a smallest subsequential limit point of the sequence.

44. Definition. Let \( \{a_n\} \) be a sequence of real numbers. The largest subsequential limit point of the sequence is denoted by \( \limsup_{n \to \infty} a_n \). The smallest subsequential limit point of the sequence is denoted by \( \liminf_{n \to \infty} a_n \).

45. Theorem. Let \( \{a_n\} \) be a sequence of real numbers, and let \( A \in \mathbb{E} \). Then \( \lim_{n \to \infty} a_n = A \) if and only if \( A = \limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n \).