Advanced Calculus I, Dr. Block, Chapter 5 notes

1. Definition. Let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$. Suppose that $a \in D$ and also a is an accumulation point of D. We say that f is differentiable at a if and only if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. In this case we denote the limit by $f'(a)$, and we call this limit the derivative of f at a.

2. Theorem. Let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$. Suppose that $a \in D$ and also a is an accumulation point of D. If f is differentiable at a, then f is continuous at a.

3. Theorem. Let $D \subseteq \mathbb{R}$, and let $f, g : D \to \mathbb{R}$. Suppose that f, g are differentiable at $x = a$. Then $f + g$, $f - g$, $f \cdot g$ are differentiable at $x = a$, and also $\frac{f}{g}$ is differentiable at $x = a$ provided that $g(a) \neq 0$, and

$$(f + g)'(a) = f'(a) + g'(a)$$

$$(f - g)'(a) = f'(a) - g'(a)$$

$$(f \cdot g)'(a) = f(a) \cdot g'(a) + g(a) \cdot f'(a)$$

$$(\frac{f}{g})'(a) = \frac{g(a) \cdot f'(a) - f(a) \cdot g'(a)}{(g(a))^2}.$$

4. Theorem. If c is a real constant, and $f(x) = c$ for all $x \in \mathbb{R}$, then $f'(x) = 0$ for all $x \in \mathbb{R}$.

If $f(x) = x$ for all $x \in \mathbb{R}$, then $f'(x) = 1$ for all $x \in \mathbb{R}$.

If n is a positive integer, and $f(x) = x^n$ for all $x \in \mathbb{R}$, then $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$.

If r is a rational number and $f(x) = x^r$ for all $x > 0$, then $f'(x) = rx^{r-1}$ for all $x > 0$.

If r is a real number and $f(x) = x^r$ for all $x > 0$, then $f'(x) = rx^{r-1}$ for all $x > 0$.

If $f(x) = \sin x$ for all $x \in \mathbb{R}$, then $f'(x) = \cos x$ for all $x \in \mathbb{R}$.

If $f(x) = \cos x$ for all $x \in \mathbb{R}$, then $f'(x) = -\sin x$ for all $x \in \mathbb{R}$.

If $f(x) = e^x$ for all $x \in \mathbb{R}$, then $f'(x) = e^x$ for all $x \in \mathbb{R}$.

If $f(x) = \ln x$ for all $x > 0$, then $f'(x) = \frac{1}{x}$ for all $x > 0$.

5. Theorem. (Chain Rule) Suppose that f, g are real valued functions, a is a real number, f is defined on some open interval containing a, and g is defined on some open interval containing $f(a)$. Suppose that f is differentiable at a and g is differentiable at $f(a)$. Then the composition $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

6. Definition. Let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$. We say that f has a relative (local) minimum at $x = c$ if and only if $c \in D$ and there exists $\delta > 0$ such that $f(c) \leq f(x)$ for all $x \in (D \cap (c - \delta, c + \delta))$.

7. Definition. Let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$. We say that f has a relative (local) maximum at $x = c$ if and only if $c \in D$ and there exists $\delta > 0$ such that $f(c) \geq f(x)$ for all $x \in (D \cap (c - \delta, c + \delta))$.

8. Definition. Let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$. We say that f has a relative (local) extremum at $x = c$ if and only if either f has a relative (local) minimum at $x = c$ or f has a relative (local) maximum at $x = c$

9. Theorem. Let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$. Suppose that f has a relative extremum at $x = c$. Also, suppose that there is an open interval (a, b) with $c \in (a, b) \subseteq D$. Finally, suppose that f is differentiable at c. Then $f'(c) = 0$.

10. Definition. Let $D \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$. We say that f is differentiable if and only if f is differentiable at a for each $a \in D$.

11. Theorem. (Inverse Function Theorem) Let I be an open interval, and suppose that $f : I \to \mathbb{R}$ is differentiable. Suppose that for each $x \in I$, $f'(x) \neq 0$. Then:

(a) $f(I)$ is an open interval and $f : I \to f(I)$ is a bijection

(b) f is either strictly increasing or strictly decreasing on I. In the first case, $f'(x) > 0$ for all $x \in I$; in the second case $f'(x) < 0$ for all $x \in I$.

(c) f^{-1} is differentiable on $f(I)$

(d) for each $y \in f(I)$ we have $(f^{-1})'(y) = \frac{1}{f'(x)}$, where $x = f^{-1}(y)$.

12. Theorem. (Rolle's Theorem) Suppose that $f : [a, b] \to \mathbb{R}$ satisfies:

(1) f is continuous on $[a, b]$.

(2) f is differentiable on (a, b).

(3) $f(a) = f(b)$.

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

13. Theorem. (Mean Value Theorem) Suppose that $f : [a, b] \to \mathbb{R}$ satisfies:

(1) f is continuous on $[a, b]$.

(2) f is differentiable on (a, b).

Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

14. Corollary. Suppose that $f : [a, b] \to \mathbb{R}$ satisfies:

(1) f is continuous on $[a, b]$.

(2) f is differentiable on (a, b).

If $f'(x) = 0$ on (a, b), then f is constant on $[a, b]$.

15. Corollary. Suppose that $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ satisfy:

(1) f and g are continuous on $[a,b]$.

(2) f and g are differentiable on (a,b).

If $f'(x) = g'(x)$ for all $x \in (a,b)$, then there is a real number k such that $f(x) - g(x) = k$ for all $x \in [a,b]$.

16. Theorem. Suppose that f is differentiable on (a,b). Then $f'(x) \geq 0$ on (a,b) if and only if f is increasing on (a,b).

17. Theorem. Suppose that f is differentiable on (a,b). Then $f'(x) \leq 0$ on (a,b) if and only if f is decreasing on (a,b).

18. Theorem. Suppose that f is differentiable on (a,b). If $f'(x) > 0$ on (a,b), then f is strictly increasing on (a,b).

19. Theorem. Suppose that f is differentiable on (a,b). If $f'(x) < 0$ on (a,b), then f is strictly decreasing on (a,b).

20. Theorem. Suppose that $f : [a,b] \to \mathbb{R}$ satisfies:

(1) f is continuous on $[a,b]$.

(2) f is differentiable on (a,b).

If $f'(x) > 0$ on (a,b), then f is strictly increasing on $[a,b]$.

21. Theorem. Suppose that $f : [a,b] \to \mathbb{R}$ satisfies:

(1) f is continuous on $[a,b]$.

(2) f is differentiable on (a,b).

If $f'(x) < 0$ on (a,b), then f is strictly decreasing on $[a,b]$.

22. Theorem. (First Derivative Test) Suppose that $a < c < b$, and f is a real valued function which is continuous on $[a,b]$. Suppose also that f is differentiable on each of the intervals (a,c) and (c,b).

(1) If $f'(x) > 0$ for all $x \in (a,c)$ and $f'(x) < 0$ for all $x \in (c,b)$, then f has a relative maximum at c.

(2) If $f'(x) < 0$ for all $x \in (a,c)$ and $f'(x) > 0$ for all $x \in (c,b)$, then f has a relative minimum at c.

23. Definition. Let $E \subseteq \mathbb{R}$, and let $f : E \to \mathbb{R}$. Let D denote the set of points $x \in E$ such that f is differentiable at x. Suppose that $a \in D$ and also a is an accumulation point of D. If the limit

$$\lim_{x \to a} \frac{f'(x) - f'(a)}{x - a}$$

exists and is finite, we denote the limit by $f''(a)$, and we call this limit the second derivative of f at a. Higher derivatives are defined inductively in the same way. We use the notation $f^{(n)}(a)$ to denote the n-th derivative of f at a.

24. Definition. The nth Taylor polynomial centered about $x = a$ is:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

25. Theorem. Suppose that f has $n+1$ derivatives in a neighborhood of a. Let $x \neq a$ be in this neighborhood. Then there exists c between x and a such that

$$f(x) = p_n(x) + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

26. Theorem. (Second Derivative Test) Suppose that f'' is continuous on an open interval containing a point c with $f'(c) = 0$.
(a) If $f''(c) > 0$, then f has a relative minimum at c.
(b) If $f''(c) < 0$, then f has a relative maximum at c.

27. Theorem. (one form of L'Hopital's rule). Suppose that f and g are differentiable on (a, b). Suppose that

$$\lim_{x \to b^-} f(x) = 0 = \lim_{x \to b^-} g(x).$$

Finally, suppose that $g'(x) \neq 0$ for all x "near" b. This means that there exists $\delta > 0$ such that $g'(x) \neq 0$ for all $x \in (b-\delta, b)$. Let $L \in \mathbb{R}$, $L = \infty$ or $L = -\infty$. If $\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to b^-} \frac{f(x)}{g(x)} = L$.

28. Theorem. (another form of L'Hopital's rule). Suppose that f and g are differentiable on (a, b). Suppose that

$$\lim_{x \to b^-} f(x) = \pm\infty = \lim_{x \to b^-} g(x).$$

Finally, suppose that $g'(x) \neq 0$ for all x "near" b. This means that there exists $\delta > 0$ such that $g'(x) \neq 0$ for all $x \in (b-\delta, b)$. Let $L \in \mathbb{R}$, $L = \infty$ or $L = -\infty$. If $\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to b^-} \frac{f(x)}{g(x)} = L$.

29. Theorem. (other forms of L'Hopital's rule). The previous two theorems are also valid for one-sided limits from the right, two-sided limits, limits as $x \to \infty$, and limits as $x \to -\infty$.

30. Remark. Note that each of the following is an indeterminate form:

$$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ 0 \cdot \infty,\ \infty - \infty,\ 1^{\infty},\ \infty^0,\ 0^0$$