1. Definitions. As a standing hypothesis, we suppose that \( f : [a, b] \to \mathbb{R} \) is a bounded function.

A partition of \([a, b]\) is a finite set of points \( P = \{x_0, x_1, \ldots, x_n\} \) with
\[
a = x_0 < x_1 < \cdots < x_n = b.
\]

A partition \( Q \) is a refinement of a partition \( P \) if and only if \( P \subseteq Q \).

Given \( f \) and \( P \) as above we define
\[
U(P, f) = \sum_{k=1}^{n} M_k \Delta x_k,
L(P, f) = \sum_{k=1}^{n} m_k \Delta x_k,
\]
where
\[
M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \quad m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\},
\]
and \( \Delta x_k = x_k - x_{k-1} \).

We call \( M_k \) an upper sum, and \( m_k \) a lower sum.

For any points \( c_1 \in [x_0, x_1], c_2 \in [x_1, x_2], \ldots, c_n \in [x_{n-1}, x_n] \) the expression
\[
S(P, f) = \sum_{k=1}^{n} f(c_k) \Delta x_k
\]
is called a Riemann sum.

2. Proposition. There exist real numbers \( m, M \) such that for any partition \( P \) of \([a, b]\) and any Riemann sum \( S(P, f) \) we have
\[
m(b - a) \leq L(P, f) \leq S(P, f) \leq U(P, f) \leq M(b - a).
\]

3. Proposition. If \( Q \) is a refinement of \( P \) then
\[
L(P, f) \leq L(Q, f) \leq U(Q, f) \leq U(P, f).
\]

4. Proposition. For any partitions \( P, Q \) of \([a, b]\) we have \( L(P, f) \leq U(Q, f) \).

5. Definition and Remark. We define the lower integral to be the supremum of all the real numbers \( L(P, f) \) for all partitions \( P \) of \([a, b]\). The lower integral is denoted by \( \int_a^b f \). We define the upper integral to be the infimum of all the real numbers \( U(P, f) \) for all partitions \( P \) of \([a, b]\). The upper integral is denoted by \( \int_a^b f \).
We say that \( f \) is Riemann integrable on \([a, b]\) if and only if the lower integral is equal to the upper integral, and denote the common value by \( \int_a^b f \).

We remark that for any partitions \( P \) and \( Q \) of \([a, b]\) we have

\[
L(P, f) \leq \int_a^b f \leq \overline{U}(Q, f).
\]

So \( f \) is Riemann integrable on \([a, b]\) if and only if \( \int_a^b f \leq \int_a^b f \).

6. Theorem. Suppose that \( f : [a, b] \to \mathbb{R} \) is a bounded function. \( f \) is Riemann integrable if and only if for every \( \epsilon > 0 \), there is a partition \( P \) such that \( U(P, f) - L(P, f) < \epsilon \).

7. Notation: We will denote the set of Riemann integrable functions \( f : [a, b] \to \mathbb{R} \) by \( R[a, b] \).

8. Theorem. If \( f : [a, b] \to \mathbb{R} \) is monotone, then \( f \in R[a, b] \).

9. Theorem. If \( f : [a, b] \to \mathbb{R} \) is continuous, then \( f \in R[a, b] \).

We remark that for the proof of Theorem 9 we need the following Definition and Theorem:

Definition: Suppose that \( D \subseteq \mathbb{R} \) and \( f : D \to \mathbb{R} \). We say that \( f \) is uniformly continuous if and only if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for every pair of points \( x, t \in D \), if \( |x - t| < \delta \) then \( |f(x) - f(t)| < \epsilon \).

Theorem: If \( f : [a, b] \to \mathbb{R} \) is continuous, then \( f \) is uniformly continuous.

10. Theorem. Let \( f : [a, b] \to \mathbb{R} \) be bounded. \( f \in R[a, b] \) if and only if there exists a real number \( A \) such that for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any Riemann sum, \( S(P, f) \), associated with a partition \( P \), whose norm is less than \( \delta \) we have \( |S(P, f) - A| < \epsilon \).

Moreover, in this case \( A = \int_a^b f \).

11. Corollary. Suppose that \( f \in R[a, b] \). Let \( \{P_n\} \) be a sequence of partitions of \([a, b]\) whose norm converges to zero. Suppose that for each positive integer \( n \), \( S(P_n, f) \) is a Riemann sum associated to the partition \( P_n \). Then

\[
\lim_{n \to \infty} S(P_n, f) = \int_a^b f.
\]

12. Theorem. Suppose that \( f, g \in R[a, b] \) and \( c \) is a real number. Then:

(a) \( f + g \in R[a, b] \), and \( \int_a^b (f + g) = \int_a^b f + \int_a^b g \).

(b) \( cf \in R[a, b] \), and \( \int_a^b (cf) = c \int_a^b f \).

13. Theorem. If \( f, g \in R[a, b] \) and \( f(x) \leq g(x) \) for all \( x \in [a, b] \), then \( \int_a^b f \leq \int_a^b g \).
14. Theorem. Suppose that \( f \in \mathbb{R}[a,b] \) and \( f([a,b]) \subseteq [c,d] \). If \( g \) is continuous on \([c,d]\), then \( g \circ f \in \mathbb{R}[a,b] \).

15. Corollary. Suppose that \( f, g \in \mathbb{R}[a,b] \). Let \( n \) be a positive integer. Then (a) \( f^n \in \mathbb{R}[a,b] \). (b) \( f \cdot g \in \mathbb{R}[a,b] \).

16. Theorem. If \( f \in \mathbb{R}[a,b] \), then \( |f| \in \mathbb{R}[a,b] \) and \( \int_a^b |f| \leq \int_a^b |f| \).

17. Theorem (one form of the mean value theorem for integrals). If \( f : [a,b] \to \mathbb{R} \) is continuous, then there exists \( c \in (a,b) \) such that
\[
\int_a^b f = f(c)(b - a).
\]

18. Theorem. Suppose that \( f : [a,b] \to \mathbb{R} \) is bounded. Then \( f \in \mathbb{R}[a,b] \) if and only if the set of points at which \( f \) is discontinuous has Lebesgue measure zero.

A set \( S \) has Lebesgue measure zero if and only if for every \( \epsilon > 0 \) there is a finite or countably infinite collection \( \{D_i\} \) of open intervals such \( S \) is a subset of the union on the \( D_i \) and the sum of the lengths of the \( D_i \) is less that \( \epsilon \).

Note that any set which is either finite or countably infinite has Lebesgue measure zero.

19. Theorem. Suppose that \( f : [a,b] \to \mathbb{R} \) is bounded, and let \( c \in (a,b) \). Then \( f \in \mathbb{R}[a,b] \) if and only if \( f \in \mathbb{R}[a,c] \) and \( f \in \mathbb{R}[c,b] \). In this case
\[
\int_a^b f = \int_a^c f + \int_c^b f.
\]

20. Theorem. (Fundamental Theorem of Calculus) Suppose that \( f : [a,b] \to \mathbb{R} \) is differentiable and \( f'(x) \in \mathbb{R}[a,b] \). Then
\[
\int_a^b f'(x)dx = f(b) - f(a).
\]

21. Theorem. Suppose that \( f \in \mathbb{R}[a,b] \). Define a function \( F : [a,b] \to \mathbb{R} \) by \( F(x) = \int_a^x f \). Then \( F \) is uniformly continuous.

22. Theorem. Suppose that \( f \in \mathbb{R}[a,b] \). Define a function \( F : [a,b] \to \mathbb{R} \) by \( F(x) = \int_a^x f \). Suppose that \( c \in [a,b] \). If \( f \) is continuous at \( c \), then \( F \) is differentiable at \( c \) and \( F'(c) = f(c) \).

23. Theorem. (Change of variables) Suppose that \( g : [c,d] \to [a,b] \) is differentiable with \( g(c) = a \) and \( g(d) = b \). Suppose also that \( g' \in \mathbb{R}[c,d] \). Finally, suppose that \( f : [a,b] \to \mathbb{R} \) is continuous. Then
\[
\int_c^d (f \circ g) \cdot g' = \int_a^b f.
\]
24. Theorem. Suppose that \( g : [c, d] \to [a, b] \) is differentiable, and \( f : [a, b] \to \mathbb{R} \) is continuous. Define \( H : [c, d] \to \mathbb{R} \) by

\[
H(x) = \int_a^{g(x)} f(t)\,dt.
\]

Then \( H \) is differentiable and \( H'(x) = (f(g(x))) \cdot g'(x) \).

25. Theorem. (Integration by parts) Suppose that \( f, g : [a, b] \to \mathbb{R} \) are differentiable and \( f', g' \in \mathbb{R}[a, b] \). Then

\[
\int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b f' \cdot g.
\]

26. Theorem. If \( f \) is continuous on \( \mathbb{R} \) and \( g \) and \( h \) are differentiable on \( \mathbb{R} \), then

\[
\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x).
\]

27. Definition. Let \( a \in \mathbb{R} \), and let \( f : [a, \infty) \to \mathbb{R} \). Suppose that \( f \) is Riemann integrable on \( [a, b] \) for each \( b > a \). If \( \lim_{b \to \infty} \int_a^b f \) exists and is some real number \( L \), then we say that the improper integral \( \int_a^\infty f \) converges to \( L \) and write \( \int_a^\infty f = L \).

28. Definition. Let \( a, b \in \mathbb{R} \) with \( a < b \), and let \( f : (a, b] \to \mathbb{R} \) be a function which is not bounded. Suppose that \( f \) is Riemann integrable on \( [c, b] \) for each \( c \) in the open interval \( (a, b) \). If \( \lim_{c \to a^+} \int_c^b f \) exists and is some real number \( L \), then we say that the improper integral \( \int_a^b f \) converges to \( L \) and write \( \int_a^b f = L \).

29. Theorem. Let \( a \in \mathbb{R} \), and let \( f : [a, \infty) \to \mathbb{R} \). Suppose that \( f \) is nonnegative, and \( f \) is Riemann integrable on \( [a, b] \) for each \( b > a \). Suppose that there exists \( M > 0 \) such that \( \int_a^b f \leq M \) for all \( b > a \). Then \( \int_a^\infty f \) converges.

30. Theorem. (Comparison test) Let \( a \in \mathbb{R} \), and let \( f, g : [a, \infty) \to \mathbb{R} \). Suppose that both \( f \) and \( g \) are Riemann integrable on \( [a, b] \) for each \( b > a \). Suppose that \( 0 \leq f(x) \leq g(x) \) for all \( x \geq a \). If \( \int_a^\infty g \) converges, then \( \int_a^\infty f \) also converges.

31. Definition. Let \( a \in \mathbb{R} \), and let \( f : [a, \infty) \to \mathbb{R} \). Suppose that \( f \) is Riemann integrable on \( [a, b] \) for each \( b > a \). We say that \( \int_a^\infty f \) converges absolutely if and only if \( \int_a^\infty |f| \) converges.

32. Theorem. Let \( a \in \mathbb{R} \), and let \( f : [a, \infty) \to \mathbb{R} \). Suppose that \( f \) is Riemann integrable on \( [a, b] \) for each \( b > a \). If \( \int_a^\infty f \) converges absolutely, then \( \int_a^\infty f \) converges.