Advanced Calculus , Dr. Block, Chapter 6 notes, 2-9-2020

1. Definitions. As a standing hypothesis, we suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function.

A partition of $[a, b]$ is a finite set of points $P = \{x_0, x_1, \ldots, x_n\}$ with

$$a = x_0 < x_1 < \cdots < x_n = b.$$

A partition Q is a refinement of a partition P if and only if $P \subseteq Q$.

Given f and P as above we define

$$U(P, f) = \sum_{k=1}^{n} M_k \Delta x_k, \;\; L(P, f) = \sum_{k=1}^{n} m_k \Delta x_k,$$

where

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \;\; m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\},$$

and $\Delta x_k = x_k - x_{k-1}$.

We call $U(P, f)$ the upper sum, and $L(P, f)$ the lower sum. Also, the maximum value of all of the Δx_k is called the norm of the partition P.

For any points $c_1 \in [x_0, x_1], c_2 \in [x_1, x_2], \ldots c_n \in [x_{n-1}, x_n]$ the expresion

$$S(P, f) = \sum_{k=1}^{n} f(c_k) \Delta x_k$$

is called a Riemann sum.

2. Proposition. There exist real numbers m, M such that for any partition P of $[a, b]$ and any Riemann sum $S(P, f)$ we have

$$m(b - a) \leq L(P, f) \leq S(P, f) \leq U(P, f) \leq M(b - a).$$

3. Proposition. If Q is a refinement of P then

$$L(P, f) \leq L(Q, f) \leq U(Q, f) \leq U(P, f).$$

4. Proposition. For any partitions P, Q of $[a, b]$ we have $L(P, f) \leq U(Q, f)$.

5. Definition and Remark. We define the lower integral to be the supremum of all the real numbers $L(P, f)$ for all partitions P of $[a, b]$. The lower integral is denoted by $\underline{\int_a^b} f$. We define the upper integral to be the infimum of all the real numbers $U(P, f)$ for all partitions P of $[a, b]$. The upper integral is denoted by $\overline{\int_a^b} f$.

We say that f is Riemann integrable on $[a, b]$ if and only if the lower integral is equal to the upper integral, and denote the common value by $\int_a^b f$.

We remark that for any partitions P and Q of $[a, b]$ we have

$$L(P, f) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(Q, f).$$

So f is Riemann integrable on $[a, b]$ if and only if $\overline{\int_a^b} f \leq \underline{\int_a^b} f$.

6. Theorem. Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function. f is Riemann integrable if and only if for every $\epsilon > 0$, there is a partition P such that

$$U(P, f) - L(P, f) < \epsilon.$$

7. Notation: We will denote the set of Riemann integrable functions $f : [a, b] \to \mathbb{R}$ by $R[a, b]$.

8. Theorem. If $f : [a, b] \to \mathbb{R}$ is monotone, then $f \in R[a, b]$.

9. Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous, then $f \in R[a, b]$.

We remark that for the proof of Theorem 9 we need the following Definition and Theorem:

Definition: Suppose that $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say that f is uniformly continuous if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every pair of points $x, t \in D$, if $|x - t| < \delta$ then $|f(x) - f(t)| < \epsilon$.

Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.

10. Theorem. Let $f : [a, b] \to \mathbb{R}$ be bounded. $f \in R[a, b]$ if and only if there exists a real number A such that for every $\epsilon > 0$, there exists $\delta > 0$ such that for any Riemann sum, $S(P, f)$, associated with a partition P, whose norm is less than δ we have $|S(P, f) - A| < \epsilon$.
Moreover, in this case $A = \int_a^b f$.

11. Corollary. Suppose that $f \in R[a, b]$. Let $\{P_n\}$ be a sequence of partitions of $[a, b]$ whose norm converges to zero. Suppose that for each positive integer n, $S(P_n, f)$ is a Riemann sum associated to the partition P_n. Then

$$\lim_{n \to \infty} S(P_n, f) = \int_a^b f.$$

12. Theorem. Suppose that $f, g \in R[a, b]$ and c is a real number. Then:
(a) $f + g \in R[a, b]$, and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.
(b) $cf \in R[a, b]$, and $\int_a^b (cf) = c \int_a^b f$.

13. Theorem. If $f, g \in R[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.

14. Theorem. Suppose that $f \in R[a,b]$ and $f([a,b]) \subseteq [c,d]$. If g is continuous on $[c,d]$, then $g \circ f \in R[a,b]$.

15. Corollary. Suppose that $f, g \in R[a,b]$. Let n be a positive integer. Then
(a) $f^n \in R[a,b]$.
(b) $f \cdot g \in R[a,b]$.

16. Theorem. If $f \in R[a,b]$, then $|f| \in R[a,b]$ and $|\int_a^b f| \leq \int_a^b |f|$.

17. Theorem (one form of the mean value theorem for integrals). If $f : [a,b] \to \mathbb{R}$ is continuous, then there exists $c \in (a,b)$ such that

$$\int_a^b f = f(c)(b-a).$$

18. Theorem. Suppose that $f : [a,b] \to \mathbb{R}$ is bounded. Then $f \in R[a,b]$ if and only if the set of points at which f is discontinuous has Lebesgue measure zero.

Definition. A set S has Lebesgue measure zero if and only if for every $\epsilon > 0$ there is a finite or countably infinite collection $\{D_i\}$ of open intervals such that S is a subset of the union on the D_i and the sum of the lengths of the D_i is less that ϵ.

Note that any set which is either finite or countably infinite has Lebesgue measure zero.

19. Theorem. Suppose that $f : [a,b] \to \mathbb{R}$ is bounded, and let $c \in (a,b)$. Then $f \in R[a,b]$ if and only if $f \in R[a,c]$ and $f \in R[c,b]$. In this case

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

20. Theorem. (Fundamental Theorem of Calculus) Suppose that $f : [a,b] \to \mathbb{R}$ is differentiable and $f'(x) \in R[a,b]$. Then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

21. Theorem. Suppose that $f \in R[a,b]$. Define a function $F : [a,b] \to \mathbb{R}$ by $F(x) = \int_a^x f$. Then F is uniformly continuous.

22. Theorem. Suppose that $f \in R[a,b]$. Define a function $F : [a,b] \to \mathbb{R}$ by $F(x) = \int_a^x f$. Suppose that $c \in [a,b]$. If f is continuous at c, then F is differentiable at c and $F'(c) = f(c)$.

23. Theorem. (Change of variables) Suppose that $g : [c,d] \to [a,b]$ is differentiable with $g(c) = a$ and $g(d) = b$. Suppose also that $g' \in R[c,d]$. Finally, suppose that $f : [a,b] \to \mathbb{R}$ is continuous. Then

$$\int_c^d (f \circ g) \cdot g' = \int_a^b f.$$

24. Theorem. Suppose that $g : [c, d] \to [a, b]$ is differentiable, and $f : [a, b] \to \mathbb{R}$ is continuous. Define $H : [c, d] \to \mathbb{R}$ by

$$H(x) = \int_a^{g(x)} f(t)dt.$$

Then H is differentiable and $H'(x) = (f(g(x))) \cdot g'(x)$.

25. Theorem. (Integration by parts) Suppose that $f, g : [a, b] \to \mathbb{R}$ are differentiable and $f', g' \in R[a, b]$. Then

$$\int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b f' \cdot g.$$

26. Theorem. If f is continuous on $\mathbb{R}$ and g and h are differentiable on $\mathbb{R}$, then

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x).$$

27. Definition. Let $a \in \mathbb{R}$, and let $f : [a, \infty) \to \mathbb{R}$. Suppose that f is Riemann integrable on $[a, b]$ for each $b > a$. If $\lim_{b \to \infty} \int_a^b f$ exists and is some real number L, then we say that the improper integral $\int_a^\infty f$ converges to L and write $\int_a^\infty f = L$.

28. Definition. Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b] \to \mathbb{R}$ be a function which is not bounded. Suppose that f is Riemann integrable on $[c, b]$ for each c in the open interval (a, b). If $\lim_{c \to a^+} \int_c^b f$ exists and is some real number L, then we say that the improper integral $\int_a^b f$ converges to L and write $\int_a^b f = L$.

29. Theorem. Let $a \in \mathbb{R}$, and let $f : [a, \infty) \to \mathbb{R}$. Suppose that f is nonnegative, and f is Riemann integrable on $[a, b]$ for each $b > a$. Suppose that there exists $M > 0$ such that $\int_a^b f \leq M$ for all $b > a$. Then $\int_a^\infty f$ converges.

30. Theorem. (Comparison test) Let $a \in \mathbb{R}$, and let $f, g : [a, \infty) \to \mathbb{R}$. Suppose that both f and g are Riemann integrable on $[a, b]$ for each $b > a$. Suppose that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. If $\int_a^\infty g$ converges, then $\int_a^\infty f$ also converges.

31. Definition. Let $a \in \mathbb{R}$, and let $f : [a, \infty) \to \mathbb{R}$. Suppose that f is Riemann integrable on $[a, b]$ for each $b > a$. We say that $\int_a^\infty f$ converges absolutely if and only if $\int_a^\infty |f|$ converges.

32. Theorem. Let $a \in \mathbb{R}$, and let $f : [a, \infty) \to \mathbb{R}$. Suppose that f is Riemann integrable on $[a, b]$ for each $b > a$. If$\int_a^\infty f$ converges absolutely, then $\int_a^\infty f$ converges.