Advanced Calculus I, Dr. Block, Chapter 2 notes

1. Theorem. (Archimedean Property) Let x be any real number. There exists a positive integer n^* greater than x.

2. Definition. A sequence is a real-valued function whose domain consists of all integers which are greater than or equal to some fixed integer (which is often 1). The notation $\{a_n\}$ is used.

3. Definition. We say that a sequence $\{a_n\}$ converges to a real number L if and only if for every $\epsilon > 0$, there exists a positive integer n^* such that for all $n \geq n^*$ we have $|a_n - L| < \epsilon$. The real number L is called the limit of the sequence and we write

$$\lim_{n\to\infty} a_n = L.$$

We also say that the sequence is convergent.

If there is no real number L as above, we say that the sequence diverges or is divergent.

4. Problem. Prove using the definition that $lim_{n\to\infty}\frac{1}{n} = 0$.

Formal Proof. Let $\epsilon > 0$. By the Archimedean Property there exists a positive integer $n^* > \frac{1}{\epsilon}$. If $n \geq n^*$ we have

$$|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{n^*} < \epsilon.$$

5. Problem. Prove using the definition that $lim_{n\to\infty}\frac{5n}{n^2+1} = 0$.

Preliminary consideration: We want $|\frac{5n}{n^2+1} - 0| < \epsilon$ for $n \geq n^*$. We see that

$$|\frac{5n}{n^2+1} - 0| = \frac{5n}{n^2+1} \leq \frac{5n}{n^2} = \frac{5}{n}.$$

Also we will have $\frac{5}{n} < \epsilon$ if $n > \frac{5}{\epsilon}$.

Formal Proof. Let $\epsilon > 0$. By the Archimedean Property there exists a positive integer $n^* > \frac{5}{\epsilon}$. If $n \geq n^*$ we have

$$|\frac{5n}{n^2+1} - 0| = \frac{5n}{n^2+1} \leq \frac{5n}{n^2} = \frac{5}{n} \leq \frac{5}{n^*} < \epsilon.$$

6. Note. A sequence $\{a_n\}$ diverges if and only if for every real number L there exists $\epsilon > 0$ such that for every positive integer n^* there exists $n \geq n^*$ with $|a_n - L| \geq \epsilon$.

7. Theorem. Any two limits of a convergent sequence are the same. (If a sequence converges, then the limit of the sequence is unique.)

8. Definition. We say that a sequence $\{a_n\}$ is bounded if and only if there is a real number B such that $|a_n| \leq B$ for all n.

9. Theorem. Any convergent sequence is bounded.

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10. Theorem. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ with $A, B \in \mathbb{R}$, then

1. $\lim_{n\to\infty} a_n + b_n = A + B$.

2. $\lim_{n\to\infty} a_n - b_n = A - B$.

3. $\lim_{n\to\infty} a_n \cdot b_n = A \cdot B$.

4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, if $B \neq 0$.

5. $\lim_{n\to\infty} (a_n)^p = A^p$, for any positive rational number p, provided that the "roots" are defined.

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11. Theorem. (Squeeze Theorem) Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences, and suppose that there exists a positive integer K such that if $n \geq K$, then $a_n \leq b_n \leq c_n$. Suppose that for some real number L

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n.$$

Then $\lim_{n\to\infty} b_n = L$.

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12. Theorem. If a sequence $\{a_n\}$ converges to 0 and a sequence $\{b_n\}$ is bounded, then the sequence $\{a_n \cdot b_n\}$ converges to 0.

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13. Theorem. (Special limits to remember and use.)

1. If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

2. If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$.

3. If $c > 0$, then $\lim_{n\to\infty} \sqrt[n]{c} = 1$.

4. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

5. If $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} \sin(a_n) = 0$.

6. If $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = 1$.

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14. Definition. We say the sequence $\{a_n\}$ diverges to ∞ if and only if for every $M > 0$, there is a positive integer n^* such that for all $n \geq n^*$ we have $a_n > M$. In this case we write

$$\lim_{n\to\infty} a_n = \infty.$$

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15. Definition. We say the sequence $\{a_n\}$ diverges to $-\infty$ if and only if for every $M < 0$, there is a positive integer n^* such that for all $n \geq n^*$ we have $a_n < M$. In this case we write

$$\lim_{n\to\infty} a_n = -\infty.$$

16. Theorem. If $\lim_{n\to\infty} a_n = \infty$ and there exists a positive integer K such that $b_n \geq a_n$ for all $n \geq K$, then $\lim_{n\to\infty} b_n = \infty$.

17. Theorem. If $\lim_{n\to\infty} a_n = -\infty$ and there exists a positive integer K such that $b_n \leq a_n$ for all $n \geq K$, then $\lim_{n\to\infty} b_n = -\infty$.

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18. Theorem. Suppose that $\lim_{n\to\infty} a_n = \infty$.

1. If $\{b_n\}$ is bounded below, then $\lim_{n\to\infty}(a_n + b_n) = \infty$.
2. If $\{b_n\}$ converges or diverges to ∞, then $\lim_{n\to\infty}(a_n + b_n) = \infty$.
3. If $\{b_n\}$ is bounded below by a positive number, then $\lim_{n\to\infty}(a_n \cdot b_n) = \infty$.
4. If $\{b_n\}$ converges to a positive number or diverges to ∞, then

$$\lim_{n\to\infty} (a_n \cdot b_n) = \infty.$$

5. If $\{b_n\}$ converges to a negative number or diverges to $-\infty$, then

$$\lim_{n\to\infty} (a_n \cdot b_n) = -\infty.$$

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19. Theorem.

1. If $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} \frac{1}{a_n} = 0$.
2. If $\lim_{n\to\infty} \frac{1}{a_n} = 0$ and $a_n > 0$ for all n sufficiently large, then

$$\lim_{n\to\infty} a_n = \infty.$$

3. If $\lim_{n\to\infty} \frac{1}{a_n} = 0$ and $a_n < 0$ for all n sufficiently large, then

$$\lim_{n\to\infty} a_n = -\infty.$$

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20. Theorem. (Ratio Test) Suppose that $\{a_n\}$ is a sequence of nonzero real numbers such that

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \alpha$$

where either $\alpha \in \mathbb{R}$ or $\alpha = \infty$.

1. If $\alpha < 1$, then $\lim_{n\to\infty} a_n = 0$.
2. If $\alpha > 1$, then $\lim_{n\to\infty} |a_n| = \infty$, so the sequence $\{a_n\}$ diverges.

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21. Definition. We say that a sequence $\{a_n\}$ oscillates if and only if none of the three statements below hold.

1. $\lim_{n\to\infty} a_n = L$ for some $L \in \mathbb{R}$.

2. $\lim_{n\to\infty} a_n = \infty$.

3. $\lim_{n\to\infty} a_n = -\infty$.

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22. Definition. We say that a sequence $\{a_n\}$ is increasing if and only if $n < k$ implies $a_n \le a_k$.

23. Remark. A sequence $\{a_n\}$ is increasing if and only if for all n we have $a_n \le a_{n+1}$.

24. Remark. A sequence $\{a_n\}$ of positive real numbers is increasing if and only if for all n we have $\frac{a_{n+1}}{a_n} \ge 1$.

25. Definition. We say that a sequence $\{a_n\}$ is eventually increasing if and only if there is a positive integer n^* such that $n^* \le n < k$ implies $a_n \le a_k$.

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26. Definition. We say that a sequence $\{a_n\}$ is decreasing if and only if $n < k$ implies $a_n \ge a_k$.

27. Remark. A sequence $\{a_n\}$ is decreasing if and only if for all n we have $a_n \ge a_{n+1}$.

28. Remark. A sequence $\{a_n\}$ of positive real numbers is decreasing if and only if for all n we have $\frac{a_{n+1}}{a_n} \le 1$.

29. Definition. We say that a sequence $\{a_n\}$ is eventually decreasing if and only if there is a positive integer n^* such that $n^* \le n < k$ implies $a_n \ge a_k$.

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30. Theorem. A bounded, increasing sequence converges. An unbounded, increasing sequence diverges to ∞.

31. Theorem. A bounded, decreasing sequence converges. An unbounded, decreasing sequence diverges to $-\infty$.

32. Definition. We say that a sequence $\{a_n\}$ is monotone if and only if either $\{a_n\}$ is increasing or $\{a_n\}$ is decreasing.

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33. Definition. Let $\epsilon > 0$, and let $s \in \mathbb{R}$. The ϵ-neighborhood of s is

$$N_\epsilon(s) = \{x \in \mathbb{R} : |x - s| < \epsilon\} = (s - \epsilon, s + \epsilon).$$

The deleted ϵ-neighborhood of s is

$$N_\epsilon^-(s) = \{x \in \mathbb{R} : 0 < |x - s| < \epsilon\} = (s - \epsilon, s) \cup (s, s + \epsilon).$$

34. Definition. Let $S \subseteq \mathbb{R}$, and let $w \in \mathbb{R}$. We say that w is an accumulation point of S if and only if every deleted neighborhood of w contains at least one point of S.

35. Theorem. Let $S \subseteq \mathbb{R}$, and let $w \in \mathbb{R}$. Then w is an accumulation point of S if and only if every neighborhood of w contains infinitely many points of S.

36. Theorem. (Bolzano-Weierstrass Theorem for sets) Every bounded infinite subset of $\mathbb{R}$ has at least one accumulation point.

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37. Definition. We say that a sequence $\{a_n\}$ is a Cauchy sequence if and only if for every $\epsilon > 0$, there exists a positive integer n^* such that for all $k, j \geq n^*$ we have $|a_k - a_j| < \epsilon$.

38. Theorem. Let $\{a_n\}$ be a sequence of real numbers. Then $\{a_n\}$ is a Cauchy sequence if and only if $\{a_n\}$ converges.

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39. Definition. The sequence $\{b_n\}_{n=i}^{\infty}$ is a subsequence of the sequence $\{a_n\}_{n=j}^{\infty}$ if and only if there exists a strictly increasing function

$$f : \{x \in \mathbb{N} : x \geq i\} \to \{x \in \mathbb{N} : x \geq j\}$$

such that $b_n = a_{f(n)}$ for all $n \in \mathbb{N}$ with $n \geq i$.

We sometimes use the notation $b_k = a_{n_k}$ for a subsequence. In this case, n_k must be a strictly increasing function of k.

40. Theorem. (Bolzano-Weierstrass Theorem for sequences) Every bounded sequence in $\mathbb{R}$ has at least one convergent subsequence.

41. Definition. We let $\mathbb{E}$ denote the set of extended real numbers defined by

$$\mathbb{E} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}.$$

42. Definition. Let $\{a_n\}$ be a sequence of real numbers, and let $A \in \mathbb{E}$. We say that A is a subsequential limit point of the sequence $\{a_n\}$ if and only if there is a subsequence a_{n_k} of $\{a_n\}$ such that

$$\lim_{k \to \infty} a_{n_k} = A.$$

43. Theorem. Let $\{a_n\}$ be a sequence of real numbers. There exists a largest subsequential limit point of the sequence and a smallest subsequential limit point of the sequence.

44. Definition. Let $\{a_n\}$ be a sequence of real numbers. The largest subsequential limit point of the sequence is denoted by $\limsup_{n \to \infty} a_n$. The smallest subsequential limit point of the sequence is denoted by $\liminf_{n \to \infty} a_n$.

45. Theorem. Let $\{a_n\}$ be a sequence of real numbers, and let $A \in \mathbb{E}$. Then $\lim_{n \to \infty} a_n = A$ if and only if $A = \limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n$.