1. Theorem. (Archimedean Property) Let \( x \) be any real number. There exists a positive integer \( n^* \) greater than \( x \).

2. Definition. A sequence is a real-valued function whose domain consists of all integers which are greater than or equal to some fixed integer (which is often 1). The notation \( \{ a_n \} \) is used.

3. Definition. We say that a sequence \( \{ a_n \} \) converges to a real number \( L \) if and only if for every \( \epsilon > 0 \), there exists a positive integer \( n^* \) such that for all \( n \geq n^* \) we have \(|a_n - L| < \epsilon\). The real number \( L \) is called the limit of the sequence and we write

\[
\lim_{n \to \infty} a_n = L.
\]

We also say that the sequence is convergent.

If there is no real number \( L \) as above, we say that the sequence diverges or is divergent.

4. Problem. Prove using the definition that \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

Formal Proof. Let \( \epsilon > 0 \). By the Archimedean Property there exists a positive integer \( n^* > \frac{1}{\epsilon} \). If \( n \geq n^* \) we have

\[
|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{n^*} < \epsilon.
\]

5. Problem. Prove using the definition that \( \lim_{n \to \infty} \frac{5n}{n^2 + 1} = 0 \).

Preliminary consideration: We want \(|\frac{5n}{n^2 + 1} - 0| < \epsilon \) for \( n \geq n^* \). We see that

\[
|\frac{5n}{n^2 + 1} - 0| = \frac{5n}{n^2 + 1} \leq \frac{5n}{n^2} = \frac{5}{n}.
\]

Also we will have \( \frac{5}{n} < \epsilon \) if \( n > \frac{5}{\epsilon} \).

Formal Proof. Let \( \epsilon > 0 \). By the Archimedean Property there exists a positive integer \( n^* > \frac{5}{\epsilon} \). If \( n \geq n^* \) we have

\[
|\frac{5n}{n^2 + 1} - 0| = \frac{5n}{n^2 + 1} \leq \frac{5n}{n^2} = \frac{5}{n} \leq \frac{5}{n^*} < \epsilon.
\]

6. Note. A sequence \( \{ a_n \} \) diverges if and only if for every real number \( L \) there exists \( \epsilon > 0 \) such that for every positive integer \( n^* \) there exists \( n \geq n^* \) with \(|a_n - L| \geq \epsilon\).

7. Theorem. Any two limits of a convergent sequence are the same. (If a sequence converges, then the limit of the sequence is unique.)

8. Definition. We say that a sequence \( \{ a_n \} \) is bounded if and only if there is a real number \( B \) such that \(|a_n| \leq B\) for all \( n \).
9. Theorem. Any convergent sequence is bounded.

10. Theorem. If \( \lim_{n \to \infty} a_n = A \) and \( \lim_{n \to \infty} b_n = B \) with \( A, B \in \mathbb{R} \), then

1. \( \lim_{n \to \infty} (a_n + b_n) = A + B \).
2. \( \lim_{n \to \infty} (a_n - b_n) = A - B \).
3. \( \lim_{n \to \infty} (a_n \cdot b_n) = A \cdot B \).
4. \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{A}{B} \), if \( B \neq 0 \).
5. \( \lim_{n \to \infty} (a_n)^p = A^p \), for any positive rational number \( p \), provided that the "roots" are defined.

11. Theorem. (Squeeze Theorem) Suppose that \( \{a_n\}, \{b_n\}, \) and \( \{c_n\} \) are sequences, and suppose that there exists a positive integer \( K \) such that if \( n \geq K \), then \( a_n \leq b_n \leq c_n \). Suppose that for some real number \( L \)

\[ \lim_{n \to \infty} a_n = L = \lim_{n \to \infty} c_n. \]

Then \( \lim_{n \to \infty} b_n = L \).

12. Theorem. If a sequence \( \{a_n\} \) converges to 0 and a sequence \( \{b_n\} \) is bounded, then the sequence \( \{a_n \cdot b_n\} \) converges to 0.

13. Theorem. (Special limits to remember and use.)

1. If \( p > 0 \), then \( \lim_{n \to \infty} \frac{1}{n^p} = 0 \).
2. If \( |r| < 1 \), then \( \lim_{n \to \infty} r^n = 0 \).
3. If \( c > 0 \), then \( \lim_{n \to \infty} \sqrt[n]{c} = 1 \).
4. \( \lim_{n \to \infty} \sqrt[n]{n} = 1 \).
5. If \( \lim_{n \to \infty} a_n = 0 \), then \( \lim_{n \to \infty} \sin(a_n) = 0 \).
6. \( \lim_{n \to \infty} a_n = 0 \), then \( \lim_{n \to \infty} \frac{\sin(a_n)}{a_n} = 1 \).

14. Definition. We say the sequence \( \{a_n\} \) diverges to \( \infty \) if and only if for every \( M > 0 \), there is a positive integer \( n^* \) such that for all \( n \geq n^* \) we have \( a_n > M \). In this case we write

\[ \lim_{n \to \infty} a_n = \infty. \]

15. Definition. We say the sequence \( \{a_n\} \) diverges to \( -\infty \) if and only if for every \( M < 0 \), there is a positive integer \( n^* \) such that for all \( n \geq n^* \) we have \( a_n < M \). In this case we write

\[ \lim_{n \to \infty} a_n = -\infty. \]
16. Theorem. If \( \lim_{n \to \infty} a_n = \infty \) and there exists a positive integer \( K \) such that \( b_n \geq a_n \) for all \( n \geq K \), then \( \lim_{n \to \infty} b_n = \infty \).

17. Theorem. If \( \lim_{n \to \infty} a_n = -\infty \) and there exists a positive integer \( K \) such that \( b_n \leq a_n \) for all \( n \geq K \), then \( \lim_{n \to \infty} b_n = -\infty \).

18. Theorem. Suppose that \( \lim_{n \to \infty} a_n = \infty \).
1. If \( \{b_n\} \) is bounded below, then \( \lim_{n \to \infty}(a_n + b_n) = \infty \).
2. If \( \{b_n\} \) converges or diverges to \( \infty \), then \( \lim_{n \to \infty}(a_n + b_n) = \infty \).
3. If \( \{b_n\} \) is bounded below by a positive number, then \( \lim_{n \to \infty}(a_n \cdot b_n) = \infty \).
4. If \( \{b_n\} \) converges to a positive number or diverges to \( \infty \), then 
   \[
   \lim_{n \to \infty} (a_n \cdot b_n) = \infty.
   \]
5. If \( \{b_n\} \) converges to a negative number or diverges to \( -\infty \), then 
   \[
   \lim_{n \to \infty} (a_n \cdot b_n) = -\infty.
   \]

19. Theorem.
1. If \( \lim_{n \to \infty} a_n = \infty \), then \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \).
2. If \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \) and \( a_n > 0 \) for all \( n \) sufficiently large, then 
   \[
   \lim_{n \to \infty} a_n = \infty.
   \]
3. If \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \) and \( a_n < 0 \) for all \( n \) sufficiently large, then 
   \[
   \lim_{n \to \infty} a_n = -\infty.
   \]

20. Theorem. (Ratio Test) Suppose that \( \{a_n\} \) is a sequence of nonzero real numbers such that 
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \alpha
\]
where either \( \alpha \in \mathbb{R} \) or \( \alpha = \infty \).
1. If \( \alpha < 1 \), then \( \lim_{n \to \infty} a_n = 0 \).
2. If \( \alpha > 1 \), then \( \lim_{n \to \infty} |a_n| = \infty \), so the sequence \( \{a_n\} \) diverges.

21. Definition. We say that a sequence \( \{a_n\} \) oscillates if and only if none of the three statements below hold.
1. \( \lim_{n \to \infty} a_n = L \) for some \( L \in \mathbb{R} \).
2. \(\lim_{n\to\infty} a_n = \infty.\)
3. \(\lim_{n\to\infty} a_n = -\infty.\)

22. Definition. We say that a sequence \(\{a_n\}\) is increasing if and only if \(n < k\) implies \(a_n \leq a_k.\)

23. Remark. A sequence \(\{a_n\}\) is increasing if and only if for all \(n\) we have \(a_n \leq a_{n+1}.\)

24. Remark. A sequence \(\{a_n\}\) of positive real numbers is increasing if and only if for all \(n\) we have \(\frac{a_{n+1}}{a_n} \geq 1.\)

25. Definition. We say that a sequence \(\{a_n\}\) is eventually increasing if and only if there is a positive integer \(n^*\) such that \(n^* \leq n < k\) implies \(a_n \leq a_k.\)

26. Definition. We say that a sequence \(\{a_n\}\) is decreasing if and only if \(n < k\) implies \(a_n \geq a_k.\)

27. Remark. A sequence \(\{a_n\}\) is decreasing if and only if for all \(n\) we have \(a_n \geq a_{n+1}.\)

28. Remark. A sequence \(\{a_n\}\) of positive real numbers is decreasing if and only if for all \(n\) we have \(\frac{a_{n+1}}{a_n} \leq 1.\)

29. Definition. We say that a sequence \(\{a_n\}\) is eventually decreasing if and only if there is a positive integer \(n^*\) such that \(n^* \leq n < k\) implies \(a_n \geq a_k.\)

30. Theorem. A bounded, increasing sequence converges. An unbounded, increasing sequence diverges to \(\infty.\)

31. Theorem. A bounded, decreasing sequence converges. An unbounded, decreasing sequence diverges to \(-\infty.\)

32. Definition. We say that a sequence \(\{a_n\}\) is monotone if and only if either \(\{a_n\}\) is increasing or \(\{a_n\}\) is decreasing.

33. Definition. Let \(\epsilon > 0,\) and let \(s \in \mathbb{R}.\) The \(\epsilon\)-neighborhood of \(s\) is
\[ N_\epsilon(s) = \{x \in \mathbb{R} : |x - s| < \epsilon\} = (s - \epsilon, s + \epsilon). \]

The deleted \(\epsilon\)-neighborhood of \(s\) is
\[ N_\epsilon^-(s) = \{x \in \mathbb{R} : 0 < |x - s| < \epsilon\} = (s - \epsilon, s) \cup (s, s + \epsilon). \]

34. Definition. Let \(S \subseteq \mathbb{R},\) and let \(w \in \mathbb{R}.\) We say that \(w\) is an accumulation point of \(S\) if and only if every deleted neighborhood of \(w\) contains at least one point of \(S.\)
35. Theorem. Let $S \subseteq \mathbb{R}$, and let $w \in \mathbb{R}$. Then $w$ is an accumulation point of $S$ if and only if every neighborhood of $w$ contains infinitely many points of $S$.

36. Theorem. (Bolzano-Weierstrass Theorem for sets) Every bounded infinite subset of $\mathbb{R}$ has at least one accumulation point.

37. Definition. We say that a sequence $\{a_n\}$ is a Cauchy sequence if and only if for every $\epsilon > 0$, there exists a positive integer $n^*$ such that for all $k, j \geq n^*$ we have $|a_k - a_j| < \epsilon$.

38. Theorem. Let $\{a_n\}$ be a sequence of real numbers. Then $\{a_n\}$ is a Cauchy sequence if and only if $\{a_n\}$ converges.

39. Definition. The sequence $\{b_n\}_{n=i}^{\infty}$ is a subsequence of the sequence $\{a_n\}_{n=j}^{\infty}$ if and only if there exists a strictly increasing function $f: \{x \in \mathbb{N}: x \geq i\} \rightarrow \{x \in \mathbb{N}: x \geq j\}$ such that $b_n = a_{f(n)}$ for all $n \in \mathbb{N}$ with $n \geq i$.

We sometimes use the notation $b_k = a_{n_k}$ for a subsequence. In this case, $n_k$ must be a strictly increasing function of $k$.

40. Theorem. (Bolzano-Weierstrass Theorem for sequences) Every bounded sequence in $\mathbb{R}$ has at least one convergent subsequence.

41. Definition. We let $\mathbb{E}$ denote the set of extended real numbers defined by $\mathbb{E} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

42. Definition. Let $\{a_n\}$ be a sequence of real numbers, and let $A \in \mathbb{E}$. We say that $A$ is a subsequential limit point of the sequence $\{a_n\}$ if and only if there is a subsequence $a_{n_k}$ of $\{a_n\}$ such that

$$\lim_{k \to \infty} a_{n_k} = A.$$

43. Theorem. Let $\{a_n\}$ be a sequence of real numbers. There exists a largest subsequential limit point of the sequence and a smallest subsequential limit point of the sequence.

44. Definition. Let $\{a_n\}$ be a sequence of real numbers. The largest subsequential limit point of the sequence is denoted by $\limsup_{n \to \infty} a_n$. The smallest subsequential limit point of the sequence is denoted by $\liminf_{n \to \infty} a_n$.

45. Theorem. Let $\{a_n\}$ be a sequence of real numbers, and let $A \in \mathbb{E}$. Then $\lim_{n \to \infty} a_n = A$ if and only if $A = \limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n$.