There are five problems worth a total of 48 points.

1. (12 points) Prove that the following sequence \( \{a_n\} \) converges to 0 using only the definition (without using any theorems).

\[
a_n = \frac{n + 9}{n^2 + 5}
\]

Solution:

Preliminary consideration: We want \( |a_n - 0| < \epsilon \) for all \( n \geq n^* \). We see that

\[
\frac{n + 9}{n^2 + 5} \leq \frac{n + 9}{n^2} \leq \frac{10n}{n^2} = \frac{10}{n}.
\]

So it suffices to have \( \frac{10}{n} < \epsilon \). This will be the case if and only if \( n > \frac{10}{\epsilon} \).

Formal proof: Let \( \epsilon > 0 \). There exists a positive integer \( n^* \) which is greater than \( \frac{10}{\epsilon} \). If \( n \geq n^* \), then

\[
|a_n - 0| = \frac{n + 9}{n^2 + 5} \leq \frac{10}{n} < \epsilon.
\]

2. (9 points) Consider two sequences \( \{a_n\} \) and \( \{b_n\} \), where the sequence \( \{a_n\} \) diverges to infinity and the sequence \( \{a_n b_n\} \) converges. Prove that the sequence \( \{b_n\} \) must converge to zero.

Solution:

Since the sequence \( \{a_n\} \) diverges to infinity, it follows that the sequence \( \left\{ \frac{1}{a_n} \right\} \) converges to zero. Now, \( b_n = \left( \frac{1}{a_n} \right)(a_n b_n) \) and the sequence \( \{a_n b_n\} \) converges to some real number \( L \). So,

\[
\lim_{n \to \infty} b_n = 0 \cdot L = 0.
\]

3. (9 points) Determine whether the given sequence converges, diverges to \( \infty \), diverges to \( -\infty \), or oscillates. If the sequence converges, find the limit. Justify your answer, using appropriate theorems.

\[
a_n = \frac{(-2)^n}{n^3}
\]

Solution:
We apply the ratio test:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{2^{n+1} \cdot n^3}{2^n \cdot (n+1)^3} = 2 > 1.$$ 

It follows from the ratio test that \(\lim_{n \to \infty} |a_n| = \infty\). Since \(a_n\) alternates between positive and negative, the sequence oscillates.

4. (9 points) Determine whether the given sequence converges, diverges to \(\infty\), diverges to \(-\infty\), or oscillates. If the sequence converges, find the limit. Justify your answer, using appropriate theorems.

\[a_n = (\sin n) \left( \frac{\sqrt{5n^4}}{n^4} \right)\]

Solution:

Since the sequence \(\{\sin n\}\) is bounded and the sequence \(\left\{\frac{1}{n}\right\}\) converges to zero, we have

$$\lim_{n \to \infty} \frac{\sin n}{n} = 0.$$ 

Also,

$$\lim_{n \to \infty} \sqrt[4]{5n^4} = \lim_{n \to \infty} \sqrt[4]{5} \cdot (\sqrt[4]{n})^4 = 1.$$ 

It follows that

$$\lim_{n \to \infty} a_n = 0.$$ 

5. (9 points) Determine whether the given sequence converges, diverges to \(\infty\), diverges to \(-\infty\), or oscillates. If the sequence converges, find the limit. Justify your answer, using appropriate theorems.

\[a_n = n^2 \sin \frac{1}{n}\]

Solution:

We observe that

$$a_n = (n) \cdot \left( \sin \frac{1}{n} \right).$$

Since \(\lim_{n \to \infty} \left( \sin \frac{1}{n} \right) = 1\), it follows that the sequence \(\{a_n\}\) diverges to infinity.