There are seven problems worth a total of 48 points.

1. (10 points) Prove that the following sequence \( \{a_n\} \) converges to 0 using only
   the definition (without using any theorems). Show your scratch work and the formal
   proof.

   \[
a_n = \frac{n + 9}{n^2 + 5}
   \]

   Solution:

   Preliminary consideration: We want \(|a_n - 0| < \epsilon\) for all \( n \geq n^* \). We see that
   \[
   \frac{n + 9}{n^2 + 5} \leq \frac{n + 9}{n^2} \leq \frac{10n}{n^2} = \frac{10}{n}.
   \]

   So it suffices to have \( \frac{10}{n} < \epsilon \). This will be the case if and only if \( n > \frac{10}{\epsilon} \).

   Formal proof: Let \( \epsilon > 0 \). There exists a positive integer \( n^* \) which is greater than
   \( \frac{10}{\epsilon} \). If \( n \geq n^* \), then
   \[
   |a_n - 0| = \frac{n + 9}{n^2 + 5} \leq \frac{10}{n} < \epsilon.
   \]

2. ( 10 points) Consider two sequences \( \{a_n\} \) and \( \{b_n\} \), where the sequence \( \{a_n\} \)
   diverges to infinity and the sequence \( \{a_n b_n\} \) converges. Prove that the sequence
   \( \{b_n\} \) must converge to zero.

   Solution:

   Since the sequence \( \{a_n\} \) diverges to infinity, it follows that the sequence \( \{\frac{1}{a_n}\} \)
   converges to zero. Now, \( b_n = (\frac{1}{a_n})(a_n b_n) \) and the sequence \( \{a_n b_n\} \) converges to
   some real number \( L \). So,
   \[
   \lim_{n \to \infty} b_n = 0 \cdot L = 0.
   \]

3. (8 points) Determine whether the given sequence converges, diverges to \( \infty \),
   diverges to \(-\infty\), or oscillates. If the sequence converges, find the limit. Justify your
   answer, using appropriate theorems.

   \[
a_n = \frac{(-2)^n}{n^3}
   \]
Solution:

We apply the ratio test:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{2^{n+1} \cdot n^3}{2^n \cdot (n + 1)^3} = 2 > 1.$$ 

It follows from the ratio test that $$\lim_{n \to \infty} |a_n| = \infty.$$ Since $$a_n$$ alternates between positive and negative, the sequence oscillates.

4. (8 points) Determine whether the given sequence converges, diverges to $$\infty$$, diverges to $$-\infty$$, or oscillates. If the sequence converges, find the limit. Justify your answer, using appropriate theorems.

$$a_n = \frac{(\sin n)(\sqrt{5n^4})}{n}$$

Solution:

Since the sequence $$\{\sin n\}$$ is bounded and the sequence $$\{\frac{1}{n}\}$$ converges to zero, we have

$$\lim_{n \to \infty} \frac{\sin n}{n} = 0.$$ 

Also,

$$\lim_{n \to \infty} \sqrt{5n^4} = \lim_{n \to \infty} \sqrt{5} \cdot (\sqrt{n})^4 = 1.$$ 

It follows that

$$\lim_{n \to \infty} a_n = 0.$$ 

5. (3 points) Determine if the statement is true or false. If $$\{a_n\}$$ is the sequence given by $$a_n = \sin \frac{n\pi}{2}$$, and $$\{b_n\}$$ is the sequence given by $$b_n = \frac{\sqrt{2}}{2}$$, then $$\{b_n\}$$ is a subsequence of $$\{a_n\}$$.

Answer: True

6. (3 points) Determine if the statement is true or false. Any sequence has at least one subsequential limit point.

Answer: True

7. (3 points) Determine if the statement is true or false. If $$S$$ is the range of the sequence $$\{a_n\}$$ given by $$a_n = \sin \frac{n\pi}{2}$$, then 0 is an accumulation point of $$S$$.

Answer: False

8. (3 points) Determine if the statement is true or false. If $$S$$ is the set of all rational numbers, then every real number is an accumulation point of $$S$$.

Answer: True