There are seven problems worth a total of 50 points.

1. (10 points) Prove that the following sequence \( \{a_n\} \) converges to 0 using only the definition (without using any theorems). Show your scratch work and the formal proof.

\[
a_n = \frac{n + 9}{n^2 + 5}
\]

Solution:

Preliminary consideration: We want \( |a_n - 0| < \epsilon \) for all \( n \geq n^* \). We see that

\[
\frac{n + 9}{n^2 + 5} \leq \frac{n + 9}{n^2} \leq \frac{10n}{n^2} = \frac{10}{n}.
\]

So it suffices to have \( \frac{10}{n} < \epsilon \). This will be the case if and only if \( n > \frac{10}{\epsilon} \).

Formal proof: Let \( \epsilon > 0 \). There exists a positive integer \( n^* \) which is greater than \( \frac{10}{\epsilon} \). If \( n \geq n^* \), then

\[
|a_n - 0| = \frac{n + 9}{n^2 + 5} \leq \frac{10}{n} < \epsilon.
\]

2. (10 points) Consider two sequences \( \{a_n\} \) and \( \{b_n\} \), where the sequence \( \{a_n\} \) diverges to infinity and the sequence \( \{a_n b_n\} \) converges. Prove that the sequence \( \{b_n\} \) must converge to zero.

Solution:

Since the sequence \( \{a_n\} \) diverges to infinity, it follows that the sequence \( \{\frac{1}{a_n}\} \) converges to zero. Now, \( b_n = (\frac{1}{a_n})(a_n b_n) \) and the sequence \( \{a_n b_n\} \) converges to some real number \( L \). So,

\[
\lim_{n \to \infty} b_n = 0 \cdot L = 0.
\]

3. (9 points) Determine whether the given sequence converges, diverges to \( \infty \), diverges to \( -\infty \), or oscillates. If the sequence converges, find the limit. Justify your answer, using appropriate theorems.

\[
a_n = \frac{(-2)^n}{n^3}
\]
Solution:

We apply the ratio test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{2^{n+1} \cdot n^3}{2^n \cdot (n+1)^3} = 2 > 1.
\]

It follows from the ratio test that \(\lim_{n \to \infty} |a_n| = \infty\). Since \(a_n\) alternates between positive and negative, the sequence oscillates.

4. (9 points) Determine whether the given sequence converges, diverges to \(\infty\), diverges to \(-\infty\), or oscillates. If the sequence converges, find the limit. Justify your answer, using appropriate theorems.

\[a_n = \frac{(\sin n)(\sqrt[4]{5n^3})}{n}\]

Solution:

Since the sequence \(\{\sin n\}\) is bounded and the sequence \(\left\{\frac{1}{n}\right\}\) converges to zero, we have

\[\lim_{n \to \infty} \frac{\sin n}{n} = 0.\]

Also,

\[\lim_{n \to \infty} \sqrt[4]{5n^3} = \lim_{n \to \infty} \sqrt[4]{5} \cdot (\sqrt[4]{n})^3 = 1.\]

It follows that

\[\lim_{n \to \infty} a_n = 0.\]

5. (3 points) Determine if the statement is true or false. If \(\{a_n\}\) is the sequence given by \(a_n = \sin \frac{n\pi}{4}\), and \(\{b_n\}\) is the sequence given by \(b_n = \frac{\sqrt{2}}{n}\), then \(\{b_n\}\) is a subsequence of \(\{a_n\}\).

Answer: True

6. (3 points) Determine if the statement is true or false. If \(\{a_n\}\) is the sequence given by \(a_n = (-1)^n \frac{2n+3}{n+7}\), then \(\lim \inf_{n \to \infty} a_n = -2.\)

Answer: True

7. (3 points) Determine if the statement is true or false. If \(S\) is the range of the sequence \(\{a_n\}\) given by \(a_n = \sin \frac{2n\pi}{7}\), then 0 is an accumulation point of \(S\).

Answer: False

8. (3 points) Determine if the statement is true or false. If \(S\) is the set of all rational numbers, then every real number is an accumulation point of \(S\).

Answer: True