ADVANCED CALCULUS I, DR. BLOCK, SAMPLE EXAM 2 (WITH SOLUTIONS), FALL 2019

There are seven problems worth a total of 50 points.

1. (10 points) Prove that the following sequence $\{a_n\}$ converges to 0 using only the definition (without using any theorems). Show your scratch work and the formal proof.

$$a_n = \frac{n+9}{n^2+5}$$

Solution:

Preliminary consideration: We want $|a_n - 0| < \epsilon$ for all $n \geq n^*$. We see that

$$\frac{n+9}{n^2+5} \leq \frac{n+9}{n^2} \leq \frac{10n}{n^2} = \frac{10}{n}.$$

So it suffices to have $\frac{10}{n} < \epsilon$. This will be the case if and only if $n > \frac{10}{\epsilon}$.

Formal proof: Let $\epsilon > 0$. There exists a positive integer n^* which is greater than $\frac{10}{\epsilon}$. If $n \geq n^*$, then

$$|a_n - 0| = \frac{n+9}{n^2+5} \leq \frac{10}{n} < \epsilon.$$

2. (10 points) Consider two sequences $\{a_n\}$ and $\{b_n\}$, where the sequence $\{a_n\}$ diverges to infinity and the sequence $\{a_n\, b_n\}$ converges. Prove that the sequence $\{b_n\}$ must converge to zero.

Solution:

Since the sequence $\{a_n\}$ diverges to infinity, it follows that the sequence $\{\frac{1}{a_n}\}$ converges to zero. Now, $b_n = (\frac{1}{a_n})(a_n\, b_n)$ and the sequence $\{a_n\, b_n\}$ converges to some real number L. So,

$$\lim_{n\to\infty} b_n = 0 \cdot L = 0.$$

3. (9 points) Determine whether the given sequence converges, diverges to ∞, diverges to $-\infty$, or oscillates. If the sequence converges, find the limit. Justify your answer, using appropriate theorems.

$$a_n = \frac{(-2)^n}{n^3}$$

Solution:

We apply the ratio test:

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{2^{n+1}\cdot n^3}{2^n\cdot(n+1)^3} = 2 > 1.$$

It follows from the ratio test that $\lim_{n\to\infty} |a_n| = \infty$. Since a_n alternates between positive and negative, the sequence oscillates.

4. (9 points) Determine whether the given sequence converges, diverges to ∞, diverges to $-\infty$, or oscillates. If the sequence converges, find the limit. Justify your answer, using appropriate theorems.

$$a_n = \frac{(\sin n)(\sqrt[n]{5n^4})}{n}$$

Solution:

Since the sequence $\{\sin n\}$ is bounded and the sequence $\{\frac{1}{n}\}$ converges to zero, we have

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Also,

$$\lim_{n\to\infty} \sqrt[n]{5n^4} = \lim_{n\to\infty} \sqrt[n]{5}\cdot(\sqrt[n]{n})^4 = 1.$$

It follows that

$$\lim_{n\to\infty} a_n = 0.$$

5. (3 points) Determine if the statement is true or false. If $\{a_n\}$ is the sequence given by $a_n = \sin\frac{n\pi}{4}$, and $\{b_n\}$ is the sequence given by $b_n = \frac{\sqrt{2}}{2}$, then $\{b_n\}$ is a subsequence of $\{a_n\}$.

Answer: True

6. (3 points) Determine if the statement is true or false. If $\{a_n\}$ is the sequence given by $a_n = (-1)^n\frac{2n+3}{n+7}$, then $\liminf_{n\to\infty} a_n = -2$.

Answer: True

7. (3 points) Determine if the statement is true or false. If S is the range of the sequence $\{a_n\}$ given by $a_n = \sin\frac{n\pi}{2}$, then 0 is an accumulation point of S.

Answer: False

8. (3 points) Determine if the statement is true or false. If S is the set of all rational numbers, then every real number is an accumulation point of S.

Answer: True