ADVANCED CALCULUS I, DR. BLOCK, SAMPLE EXAM 1 WITH SOLUTIONS, FALL 2019

There are 7 problems worth a total of 50 points.

1. (10 points) Use mathematical induction to prove the given statement.

For every positive integer n,

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

Solution:

First, we prove that the statement is true for $n = 1$. For $n = 1$, each side is equal to 1, so the statement is true.

Second, we suppose that the statement is true for $n = k$, for some integer $k \geq 1$. We must show the statement is true for $n = k + 1$. So, we are given

$$\sum_{j=1}^{k}(2j-1) = k^2.$$

We have

$$\sum_{j=1}^{k+1}(2j-1) = (\sum_{j=1}^{k}(2j-1)) + (2(k+1)-1) = k^2 + 2k + 1 = (k+1)^2.$$

This shows that the statement is true for $n = k + 1$ as desired. $\square$

2. (8 points) Negate the statement: There exists a real number b such that $f(x) \leq b$ for all $x \in D$.

Solution: For every real number b, there exists $x \in D$ with $f(x) > b$.

3. (10 points) Find all real values of x that satisy the given expression. Express your answer as an interval on the real line, a union of intervals, a finite set of real numbers, or the empty set. Show your work.

$$|2x-5| \leq |x+4|.$$

Solution: Since both sides of the inequality are non-negative we obtain an equivalent inequality by squaring both sides. By bringing all of the terms to the left side and factoring we obtain the equivalent inequality:

$$(3x-1)(x-9) \le 0$$

So the set of real numbers which satisy the inequality is $[\frac{1}{3}, 9]$.

4. (10 points) Prove the following: If $|f(x)| \le M$ for all $x \in [a,b]$, then

$$-2M \le f(x_1) - f(x_2) \le 2M$$

for any $x_1, x_2 \in [a,b]$.

Solution: Suppose that $|f(x)| \le M$ for all $x \in [a,b]$. Suppose that $x_1, x_2 \in [a,b]$. We have

$$-M \le f(x_1) \le M$$

and

$$-M \le -f(x_2) \le M.$$

Adding, we obtain

$$-2M \le f(x_1) - f(x_2) \le 2M.$$

5. (4 points) Determine if the statement is true or false.

If A and B are sets, then

$$(A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B).$$

Solution: The statement is true. Although you are not expected to give any proofs in the true false questions on the exam, we include a proof here.

First we show that

$$(A \backslash B) \cup (B \backslash A) \subseteq (A \cup B) \backslash (A \cap B).$$

Let $x \in (A \backslash B) \cup (B \backslash A)$. Then either $x \in (A \backslash B)$ or $x \in (B \backslash A)$.

Case 1. $x \in (A \backslash B)$.

Then $x \in A$ and $x \notin B$. It follows that $x \in (A \cup B)$ and $x \notin (A \cap B)$. Hence $x \in (A \cup B) \backslash (A \cap B)$.

Case 2. $x \in (B \backslash A)$.

Then $x \in B$ and $x \notin A$. It follows that $x \in (A \cup B)$ and $x \notin (A \cap B)$. Hence $x \in (A \cup B) \backslash (A \cap B)$.

Second, we show that

$$(A \cup B) \backslash (A \cap B) \subseteq (A \backslash B) \cup (B \backslash A).$$

Let $x \in (A \cup B) \backslash (A \cap B)$. Then x is in one of the sets A, B but not both. So either $x \in (A \backslash B)$ or $x \in (B \backslash A)$. It follows that $x \in (A \backslash B) \cup (B \backslash A)$. $\square$

6. (4 points). Determine if the statement is true or false.

If $f : X \to Y$ and $A \subseteq X$, then

$$f^{-1}(f(A)) = A.$$

Solution: The statement is false.
Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Set $A = [0, 1]$. Then $f(A) = [0, 1]$ and

$$f^{-1}(f(A)) = f^{-1}([0, 1]) = [-1, 1].$$

7. (4 points). Determine if the statement is true or false.

If $S \subseteq \mathbb{R}$ and k is the supremum of S, then $k \in S$.

Solution: The statement is false.
Let S be the open interval $(0, 1)$, and let $k = 1$. Then k is the supremum of S, and $k \notin S$.