There are 7 problems worth a total of 50 points.

1. (10 points) Use mathematical induction to prove the given statement. 
For every positive integer \( n \),
\[
\sum_{k=1}^{n} (2k - 1) = n^2.
\]

Solution:
First, we prove that the statement is true for \( n = 1 \). For \( n = 1 \), each side is equal to 1, so the statement is true.

Second, we suppose that the statement is true for \( n = k \), for some integer \( k \geq 1 \). We must show the statement is true for \( n = k + 1 \). So, we are given
\[
\sum_{j=1}^{k} (2j - 1) = k^2.
\]

We have
\[
\sum_{j=1}^{k+1} (2j - 1) = \left( \sum_{j=1}^{k} (2j - 1) \right) + (2(k + 1) - 1) = k^2 + 2k + 1 = (k + 1)^2.
\]

This shows that the statement is true for \( n = k + 1 \) as desired. \( \square \)

2. (8 points) Negate the statement: There exists a real number \( b \) such that 
\( f(x) \leq b \) for all \( x \in D \).

Solution: For every real number \( b \), there exists \( x \in D \) with \( f(x) > b \).

3. (10 points) Find all real values of \( x \) that satisfy the given expression. Express your answer as an interval on the real line, a union of intervals, a finite set of real numbers, or the empty set. Show your work.
\[
|2x - 5| \leq |x + 4|.
\]

Solution: Since both sides of the inequality are non-negative we obtain an equivalent inequality by squaring both sides. By bringing all of the terms to the left side and factoring we obtain the equivalent inequality:
So the set of real numbers which satisfy the inequality is $[\frac{1}{3}, 9]$.

4. (10 points) Prove the following: If $|f(x)| \leq M$ for all $x \in [a, b]$, then

$$-2M \leq f(x_1) - f(x_2) \leq 2M$$

for any $x_1, x_2 \in [a, b]$.

Solution: Suppose that $|f(x)| \leq M$ for all $x \in [a, b]$. Suppose that $x_1, x_2 \in [a, b]$.

We have

$$-M \leq f(x_1) \leq M$$

and

$$-M \leq -f(x_2) \leq M.$$

Adding, we obtain

$$-2M \leq f(x_1) - f(x_2) \leq 2M.$$

5. (4 points) Determine if the statement is true or false.

If $A$ and $B$ are sets, then

$$(A\setminus B) \cup (B\setminus A) = (A \cup B)\setminus (A \cap B).$$

Solution: The statement is true. Although you are not expected to give any proofs in the true false questions on the exam, we include a proof here.

First we show that

$$(A\setminus B) \cup (B\setminus A) \subseteq (A \cup B)\setminus (A \cap B).$$

Let $x \in (A\setminus B) \cup (B\setminus A)$. Then either $x \in (A\setminus B)$ or $x \in (B\setminus A)$.

Case 1. $x \in (A\setminus B)$.

Then $x \in A$ and $x \notin B$. It follows that $x \in (A \cup B)$ and $x \notin (A \cap B)$. Hence $x \in (A \cup B)\setminus (A \cap B)$.

Case 2. $x \in (B\setminus A)$.

Then $x \in B$ and $x \notin A$. It follows that $x \in (A \cup B)$ and $x \notin (A \cap B)$. Hence $x \in (A \cup B)\setminus (A \cap B)$.

Second, we show that

$$(A \cup B)\setminus (A \cap B) \subseteq (A\setminus B) \cup (B\setminus A).$$

Let $x \in (A \cup B)\setminus (A \cap B)$. Then $x$ is in one of the sets $A, B$ but not both. So either $x \in (A\setminus B)$ or $x \in (B\setminus A)$. It follows that $x \in (A\setminus B) \cup (B\setminus A)$. □

6. (4 points). Determine if the statement is true or false.
If $f : X \to Y$ and $A \subseteq X$, then
\[ f^{-1}(f(A)) = A. \]

Solution: The statement is false. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Set $A = [0, 1]$. Then $f(A) = [0, 1]$ and
\[ f^{-1}(f(A)) = f^{-1}([0, 1]) = [-1, 1]. \]

7. (4 points). Determine if the statement is true or false.
If $S \subseteq \mathbb{R}$ and $k$ is the supremum of $S$, then $k \in S$.

Solution: The statement is false. Let $S$ be the open interval $(0, 1)$, and let $k = 1$. Then $k$ is the supremum of $S$, and $k \notin S$. 