Advanced Calculus, Dr. Block, Proofs for Exam 2

1. If \( f : [a, b] \to \mathbb{R} \) is continuous, then \( f \in R[a, b] \).

2. Suppose that \( f, g \in R[a, b] \). Let \( n \) be a positive integer. Then
   (a) \( f^n \in R[a, b] \).
   (b) \( f \cdot g \in R[a, b] \).

3. Theorem. If \( f \in R[a, b] \), then \( |f| \in R[a, b] \) and \( |\int_{a}^{b} f| \leq \int_{a}^{b} |f| \).

4. (Fundamental Theorem of Calculus) Suppose that \( f : [a, b] \to \mathbb{R} \) is differentiable and \( f'(x) \in R[a, b] \). Then
   \[
   \int_{a}^{b} f'(x) dx = f(b) - f(a).
   \]

5. Theorem. Suppose that \( f \in R[a, b] \). Define a function \( F : [a, b] \to \mathbb{R} \) by \( F(x) = \int_{a}^{x} f \). Then \( F \) is uniformly continuous.

6. Theorem. (Change of variables) Suppose that \( g : [c, d] \to [a, b] \) is differentiable with \( g(c) = a \) and \( g(d) = b \). Suppose also that \( g' \in R[c, d] \). Finally, suppose that \( f : [a, b] \to \mathbb{R} \) is continuous. Then
   \[
   \int_{c}^{d} (f \circ g) \cdot g' = \int_{a}^{b} f.
   \]