6. This means that for every natural number $x$, the statement $\forall y(x < y)$ is true. But as we saw in the third statement, there isn’t even one value of $x$ for which this statement is true. Thus, $\forall x \forall y(x < y)$ is false.

Exercises

*1. Analyze the logical forms of the following statements.
   (a) Anyone who has forgiven at least one person is a saint.
   (b) Nobody in the calculus class is smarter than everybody in the discrete math class.
   (c) Everyone likes Mary, except Mary herself.
   (d) Jane saw a police officer, and Roger saw one too.
   (e) Jane saw a police officer, and Roger saw him too.

2. Analyze the logical forms of the following statements.
   (a) Anyone who has bought a Rolls Royce with cash must have a rich uncle.
   (b) If anyone in the dorm has the measles, then everyone who has a friend in the dorm will have to be quarantined.
   (c) If nobody failed the test, then everybody who got an A will tutor someone who got a D.
   (d) If anyone can do it, Jones can.
   (e) If Jones can do it, anyone can.

3. Analyze the logical forms of the following statements. The universe of discourse is $\mathbb{R}$. What are the free variables in each statement?
   (a) Every number that is larger than $x$ is larger than $y$.
   (b) For every number $a$, the equation $ax^2 + 4x - 2 = 0$ has at least one solution iff $a \geq -2$.
   (c) All solutions of the inequality $x^3 - 3x < 3$ are smaller than 10.
   (d) If there is a number $x$ such that $x^2 + 5x = w$ and there is a number $y$ such that $4 - y^3 = w$, then $w$ is between $-10$ and 10.

*4. Translate the following statements into idiomatic English.
   (a) $\forall x((H(x) \land \lnot \exists y M(x, y)) \rightarrow U(x))$, where $H(x)$ means “$x$ is a man,” $M(x, y)$ means “$x$ is married to $y$,” and $U(x)$ means “$x$ is unhappy.”
   (b) $\exists z (P(z, x) \land S(z, y) \land W(y))$, where $P(z, x)$ means “$z$ is a parent of $x$,” $S(z, y)$ means “$z$ and $y$ are siblings,” and $W(y)$ means “$y$ is a woman.”

5. Translate the following statements into idiomatic mathematical English.
   (a) $\forall x((P(x) \land \lnot(x = 2)) \rightarrow O(x))$, where $P(x)$ means “$x$ is a prime number” and $O(x)$ means “$x$ is odd.”
(b) $\exists x[P(x) \land \forall y(P(y) \rightarrow y \leq x)]$, where $P(x)$ means "$x$ is a perfect number."

6. Are these statements true or false? The universe of discourse is the set of all people, and $P(x, y)$ means "$x$ is a parent of $y$.”
   (a) $\exists x \forall y P(x, y)$.
   (b) $\forall x \exists y P(x, y)$.
   (c) $\neg \exists x \exists y P(x, y)$.
   (d) $\exists x \neg \exists y P(x, y)$.
   (e) $\exists x \exists y \neg P(x, y)$.

7. Are these statements true or false? The universe of discourse is $\mathbb{N}$.
   (a) $\forall x \exists y(2x - y = 0)$.
   (b) $\exists y \forall x(2x - y = 0)$.
   (c) $\forall x \exists y(x - 2y = 0)$.
   (d) $\forall x(x < 10 \rightarrow \forall y(y < x \rightarrow y < 9))$.
   (e) $\exists y \exists z(y + z = 100)$.
   (f) $\forall x \exists y(y > x \land \exists z(y + z = 100))$.

8. Same as exercise 7 but with $\mathbb{R}$ as the universe of discourse.
9. Same as exercise 7 but with $\mathbb{Z}$ as the universe of discourse.

2.2. Equivalences Involving Quantifiers

In our study of logical connectives in Chapter 1 we found it useful to examine equivalences between different formulas. In this section, we will see that there are also a number of important equivalences involving quantifiers.

For example, in Example 2.1.2 we represented the statement "Nobody’s perfect” by the formula $\neg \exists x P(x)$, where $P(x)$ meant “$x$ is perfect.” But another way to express the same idea would be to say that everyone fails to be perfect, or in other words $\forall x \neg P(x)$. This suggests that these two formulas are equivalent, and a little thought should show that they are. No matter what $P(x)$ stands for, the formula $\neg \exists x P(x)$ means that there’s no value of $x$ in the universe of discourse for which $P(x)$ is true. But that’s the same as saying that for every value of $x$ in the universe, $P(x)$ is false, or in other words $\forall x \neg P(x)$. Thus, $\neg \exists x P(x)$ is equivalent to $\forall x \neg P(x)$.

Similar reasoning shows that $\neg \forall x P(x)$ is equivalent to $\exists x \neg P(x)$. To say that $\neg \forall x P(x)$ means that it is not the case that for all values of $x$, $P(x)$ is true. That’s equivalent to saying there’s at least one value of $x$ for which $P(x)$ is false, which is what it means to say $\exists x \neg P(x)$. For example, in Example 2.1.2 we translated “Someone didn’t do the homework” as $\exists x \neg H(x)$, where $H(x)$ stands for “$x$ did the homework.” An equivalent statement would be “Not everyone did the homework,” which would be represented by the formula $\neg \forall x H(x)$. 