72  Quantificational Logic

(ii) $\exists a(x = ya) \land \exists b(x = zb) \land \neg \exists w(w < x \land (w \text{ is a multiple of both } y \text{ and } z))$.

(iii) $\exists a(x = ya) \land \exists b(x = zb) \land \neg \exists w(w < x \land \exists c(w = yc) \land \exists d(w = zd))$.

2. (a) $\forall x(x + 0 = x)$.
(b) $\forall x \exists y(x + y = 0)$.
(c) $\forall x(x < 0 \rightarrow \neg \exists y(y^2 = x))$.
(d) We translate this gradually:

(i) $\forall x(x > 0 \rightarrow x \text{ has exactly two square roots})$.
(ii) $\forall x(x > 0 \rightarrow \exists y \exists z(y \text{ and } z \text{ are square roots of } x \text{ and } y \neq z \text{ and nothing else is a square root of } x)$.
(iii) $\forall x(x > 0 \rightarrow \exists y \exists z(y^2 = x \land z^2 = x \land y \neq z \land \neg \exists w(w^2 = x \land w \neq y \land w \neq z))$.

Exercises

1. Negate these statements and then reexpress the results as equivalent positive statements. (See Example 2.2.1.)
(a) Everyone who is majoring in math has a friend who needs help with his homework.
(b) Everyone has a roommate who dislikes everyone.
(c) $A \cup B \subseteq C \setminus D$.
(d) $\exists x \forall y(y > x \rightarrow \exists z(z^2 + 5z = y))$.

2. Negate these statements and then reexpress the results as equivalent positive statements. (See Example 2.2.1.)
(a) There is someone in the freshman class who doesn’t have a roommate.
(b) Everyone likes someone, but no one likes everyone.
(c) $\forall a \in A \exists b \in B(a \in C \leftrightarrow b \in C)$.
(d) $\forall y > 0 \exists x(ax^2 + bx + c = y)$.

3. Are these statements true or false? The universe of discourse is $\mathbb{N}$.
(a) $\forall x(x < 7 \rightarrow \exists a \exists b \exists c(a^2 + b^2 + c^2 = x))$.
(b) $\exists x((x - 4)^2 = 9)$.
(c) $\exists x((x - 4)^2 = 25)$.
(d) $\exists x \exists y((x - 4)^2 = 25 \land (y - 4)^2 = 25)$.

4. Show that the second quantifier negation law, which says that $\neg \forall x P(x)$ is equivalent to $\exists x \neg P(x)$, can be derived from the first, which says that $\neg \exists x P(x)$ is equivalent to $\forall x \neg P(x)$. (Hint: Use the double negation law.)

5. Show that $\neg \exists x \in A P(x)$ is equivalent to $\forall x \in A \neg P(x)$.

6. Show that the existential quantifier distributes over disjunction. In other words, show that $\exists x (P(x) \lor Q(x))$ is equivalent to $\exists x P(x) \lor \exists x Q(x)$.
More Operations on Sets

(Hint: Use the fact, discussed in this section, that the universal quantifier distributes over conjunction.)

7. Show that \( \exists x(P(x) \rightarrow Q(x)) \) is equivalent to \( \forall x P(x) \rightarrow \exists x Q(x) \).

8. Show that \( (\forall x \in A P(x)) \land (\forall x \in B P(x)) \) is equivalent to \( \forall x \in (A \cup B) P(x) \). (Hint: Start by writing out the meanings of the bounded quantifiers in terms of unbounded quantifiers.)

9. Is \( \forall x (P(x) \lor Q(x)) \) equivalent to \( \forall x P(x) \lor \forall x Q(x) \)? Explain. (Hint: Try assigning meanings to \( P(x) \) and \( Q(x) \).)

10. (a) Show that \( \exists x \in A P(x) \lor \exists x \in B P(x) \) is equivalent to \( \exists x \in (A \cup B) P(x) \).

(b) Is \( \exists x \in A P(x) \land \exists x \in B P(x) \) equivalent to \( \exists x \in (A \cap B) P(x) \)? Explain.

11. Show that the statements \( A \subseteq B \) and \( A \setminus B = \emptyset \) are equivalent by writing each in logical symbols and then showing that the resulting formulas are equivalent.

12. Let \( T(x, y) \) mean "\( x \) is a teacher of \( y \)." What do the following statements mean? Under what circumstances would each one be true? Are any of them equivalent to each other?

(a) \( \exists y T(x, y) \).

(b) \( \exists x \exists y T(x, y) \).

(c) \( \exists y \exists x T(x, y) \).

(d) \( \exists y \exists y T(x, y) \).

(e) \( \exists y \exists y T(x, y) \).

(f) \( \exists y \exists y (T(x, y) \land \neg \exists u \forall v (T(u, v) \land (u \neq x \lor v \neq y))) \).

2.3. More Operations on Sets

Now that we know how to work with quantifiers, we are ready to discuss some more advanced topics in set theory.

So far, the only way we have to define sets, other than listing their elements one by one, is to use the elementhood test notation \( \{ x \mid P(x) \} \). Sometimes this notation is modified by allowing the \( x \) before the vertical line to be replaced with a more complex expression. For example, suppose we wanted to define \( S \) to be the set of all perfect squares. Perhaps the easiest way to describe this set is to say that it consists of all numbers of the form \( n^2 \), where \( n \) is a natural number. This is written \( S = \{ n^2 \mid n \in \mathbb{N} \} \). Note that, using our solution for the first statement from Example 2.2.3, we could also define this set by writing \( S = \{ x \mid \exists n \in \mathbb{N} (x = n^2) \} \). Thus, \( \{ n^2 \mid n \in \mathbb{N} \} = \{ x \mid \exists n \in \mathbb{N} (x = n^2) \} \), and therefore \( x \in \{ n^2 \mid n \in \mathbb{N} \} \) means the same thing as \( \exists n \in \mathbb{N} (x = n^2) \).