Proofs

2. Prove that if \( A \) and \( B \setminus C \) are disjoint, then \( A \cap B \subseteq C \).

3. Prove that if \( A \subseteq B \setminus C \) then \( A \) and \( C \) are disjoint.

4. Suppose \( A \subseteq \mathcal{P}(A) \). Prove that \( \mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A)) \).

5. The hypothesis of the theorem proven in exercise 4 is \( A \subseteq \mathcal{P}(A) \).
   (a) Can you think of a set \( A \) for which this hypothesis is true?
   (b) Can you think of another?

6. Suppose \( x \) is a real number.
   (a) Prove that if \( x \neq 1 \) then there is a real number \( y \) such that \( \frac{x+1}{y-2} = x \).
   (b) Prove that if there is a real number \( y \) such that \( \frac{x+1}{y-2} = x \), then \( x \neq 1 \).

7. Prove that for every real number \( x \), if \( x > 2 \) then there is a real number \( y \) such that \( y + \frac{1}{y} = x \).

8. Prove that if \( \mathcal{F} \) is a family of sets and \( A \in \mathcal{F} \), then \( A \subseteq \bigcup \mathcal{F} \).

9. Prove that if \( \mathcal{F} \) is a family of sets and \( A \in \mathcal{F} \), then \( \bigcap \mathcal{F} \subseteq A \).

10. Suppose that \( \mathcal{F} \) is a nonempty family of sets, \( B \) is a set, and \( \forall A \in \mathcal{F} (B \subseteq A) \). Prove that \( B \subseteq \bigcap \mathcal{F} \).

11. Suppose that \( \mathcal{F} \) is a family of sets. Prove that if \( \emptyset \in \mathcal{F} \), then \( \bigcap \mathcal{F} = \emptyset \).

12. Suppose \( \mathcal{F} \) and \( \mathcal{G} \) are families of sets. Prove that if \( \mathcal{F} \subseteq \mathcal{G} \) then \( \bigcup \mathcal{F} \subseteq \bigcup \mathcal{G} \).

13. Suppose \( \mathcal{F} \) and \( \mathcal{G} \) are nonempty families of sets. Prove that if \( \mathcal{F} \subseteq \mathcal{G} \) then \( \bigcap \mathcal{G} \subseteq \bigcap \mathcal{F} \).

14. Suppose \( \{ A_i \mid i \in I \} \) is an indexed family of sets. Prove that \( \bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i) \). (Hint: First make sure you know what all the notation means!)

15. Suppose \( \{ A_i \mid i \in I \} \) is an indexed family of sets and \( I \neq \emptyset \). Prove that \( \bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathcal{P}(A_i) \).

16. Prove the converse of the statement proven in Example 3.3.5. In other words, prove that if \( \mathcal{F} \subseteq \mathcal{P}(B) \) then \( \bigcup \mathcal{F} \subseteq B \).

17. Suppose \( \mathcal{F} \) and \( \mathcal{G} \) are nonempty families of sets, and every element of \( \mathcal{F} \) is a subset of every element of \( \mathcal{G} \). Prove that \( \bigcup \mathcal{F} \subseteq \bigcap \mathcal{G} \).

18. In this problem all variables range over \( \mathbb{Z} \), the set of all integers.
   (a) Prove that if \( a \mid b \) and \( a \mid c \), then \( a \mid (b + c) \).
   (b) Prove that if \( ac \mid bc \) and \( c \neq 0 \), then \( a \mid b \).

19. (a) Prove that for all real numbers \( x \) and \( y \) there is a real number \( z \) such that \( x + z = y - z \).
   (b) Would the statement in part (a) be correct if "real number" were changed to "integer"? Justify your answer.

20. Consider the following theorem:

   **Theorem.** For every real number \( x \), \( x^2 \geq 0 \).

   What's wrong with the following proof of the theorem?
Proof. Suppose not. Then for every real number \(x\), \(x^2 < 0\). In particular, plugging in \(x = 3\) we would get \(9 < 0\), which is clearly false. This contradiction shows that for every number \(x\), \(x^2 \geq 0\). \(\square\)

21. Consider the following incorrect theorem:

**Incorrect Theorem.** If \(\forall x \in A(x \neq 0)\) and \(A \subseteq B\) then \(\forall x \in B(x \neq 0)\).

(a) What's wrong with the following proof of the theorem?

*Proof.* Let \(x\) be an arbitrary element of \(A\). Since \(\forall x \in A(x \neq 0)\), we can conclude that \(x \neq 0\). Also, since \(A \subseteq B\), \(x \in B\). Since \(x \in B\), \(x \neq 0\), and \(x\) was arbitrary, we can conclude that \(\forall x \in B(x \neq 0)\). \(\square\)

(b) Find a counterexample to the theorem. In other words, find an example of sets \(A\) and \(B\) for which the hypotheses of the theorem are true but the conclusion is false.

*22. Consider the following incorrect theorem:

**Incorrect Theorem.** \(\exists x \in \mathbb{R} \forall y \in \mathbb{R}(xy^2 = y - x)\).

What's wrong with the following proof of the theorem?

*Proof.* Let \(x = y/(y^2 + 1)\). Then

\[
y - x = y - \frac{y}{y^2 + 1} = \frac{y^3}{y^2 + 1} = \frac{y}{y^2 + 1} \cdot y^2 = xy^2.
\]

\(\square\)

23. Consider the following incorrect theorem:

**Incorrect Theorem.** Suppose \(\mathcal{F}\) and \(\mathcal{G}\) are families of sets. If \(\cup \mathcal{F}\) and \(\cup \mathcal{G}\) are disjoint, then so are \(\mathcal{F}\) and \(\mathcal{G}\).

(a) What's wrong with the following proof of the theorem?

*Proof.* Suppose \(\cup \mathcal{F}\) and \(\cup \mathcal{G}\) are disjoint. Suppose \(\mathcal{F}\) and \(\mathcal{G}\) are not disjoint. Then we can choose some set \(A\) such that \(A \in \mathcal{F}\) and \(A \in \mathcal{G}\). Since \(A \in \mathcal{F}\), by exercise 8, \(A \subseteq \cup \mathcal{F}\), so every element of \(A\) is in \(\cup \mathcal{F}\). Similarly, since \(A \in \mathcal{G}\), every element of \(A\) is in \(\cup \mathcal{G}\). But then every element of \(A\) is in both \(\cup \mathcal{F}\) and \(\cup \mathcal{G}\), and this is impossible since \(\cup \mathcal{F}\) and \(\cup \mathcal{G}\) are disjoint. Thus, we have reached a contradiction, so \(\mathcal{F}\) and \(\mathcal{G}\) must be disjoint. \(\square\)

(b) Find a counterexample to the theorem.
24. Consider the following putative theorem:

**Theorem?** For all real numbers \( x \) and \( y \), \( x^2 + xy - 2y^2 = 0 \).

(a) What's wrong with the following proof of the theorem?

**Proof.** Let \( x \) and \( y \) be equal to some arbitrary real number \( r \). Then

\[
x^2 + xy - 2y^2 = r^2 + r \cdot r - 2r^2 = 0.
\]

Since \( x \) and \( y \) were both arbitrary, this shows that for all real numbers \( x \) and \( y \), \( x^2 + xy - 2y^2 = 0 \). \( \Box \)

(b) Is the theorem correct? Justify your answer with either a proof or a counterexample.

*25. Prove that for every real number \( x \) there is a real number \( y \) such that for every real number \( z \), \( yz = (x + z)^2 - (x^2 + z^2) \).

26. (a) Comparing the various rules for dealing with quantifiers in proofs, you should see a similarity between the rules for goals of the form \( \forall x P(x) \) and givens of the form \( \exists x P(x) \). What is this similarity? What about the rules for goals of the form \( \exists x P(x) \) and givens of the form \( \forall x P(x) \)?

(b) Can you think of a reason why these similarities might be expected?

(Hint: Think about how proof by contradiction works when the goal starts with a quantifier.)

### 3.4. Proofs Involving Conjunctions and Biconditionals

The method for proving a goal of the form \( P \land Q \) is so simple it hardly seems worth mentioning:

**To prove a goal of the form \( P \land Q \):**

Prove \( P \) and \( Q \) separately.

In other words, a goal of the form \( P \land Q \) is treated as two separate goals: \( P \) and \( Q \). The same is true of givens of the form \( P \land Q \):

**To use a given of the form \( P \land Q \):**

Treat this given as two separate givens: \( P \), and \( Q \).

We've already used these ideas, without mention, in some of our previous examples. For example, the definition of the given \( x \in A \setminus C \) in Example 3.2.3 was \( x \in A \land x \notin C \), but we treated it as two separate givens: \( x \in A \), and \( x \notin C \).