the integer $k$ is justified by existential instantiation, since the assumption $6 \mid n$ means $\exists k \in \mathbb{Z} (6k = n)$. At this point in the proof we have the following givens and goals:

<table>
<thead>
<tr>
<th>Givens</th>
<th>Goals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \in \mathbb{Z}$</td>
<td>$2 \mid n$</td>
</tr>
<tr>
<td>$k \in \mathbb{Z}$</td>
<td>$3 \mid n$</td>
</tr>
<tr>
<td>$6k = n$</td>
<td></td>
</tr>
</tbody>
</table>

The first goal, $2 \mid n$, means $\exists j \in \mathbb{Z} (2j = n)$, so we must find an integer $j$ such that $2j = n$. Although the proof doesn’t say so explicitly, the equation $n = 2(3k)$, which is derived in the proof, suggests that the value being used for $j$ is $j = 3k$. Clearly, $3k$ is an integer (another step skipped in the proof), so this choice for $j$ works. The proof of $3 \mid n$ is similar.

For the right-to-left direction we assume $2 \mid n$ and $3 \mid n$ and prove $6 \mid n$. Once again, the introduction of $j$ and $k$ is justified by existential instantiation. No explanation is given for why we should compute $6(j - k)$, but a proof need not provide such explanations. The reason for the calculation should become clear when, surprisingly, it turns out that $6(j - k) = n$. Such surprises provide part of the pleasure of working with proofs. As in the first half of the proof, since $j - k$ is an integer, this shows that $6 \mid n$.

**Exercises**

*1. Use the methods of this chapter to prove that $\forall x (P(x) \land Q(x))$ is equivalent to $\forall x P(x) \land \forall x Q(x)$.

*2. Prove that if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

*3. Suppose $A \subseteq B$. Prove that for every set $C$, $C \setminus B \subseteq C \setminus A$.

*4. Prove that if $A \subseteq B$ and $A \not\subseteq C$ then $B \not\subseteq C$.

*5. Prove that if $A \subseteq B \setminus C$ and $A \not\subseteq \emptyset$ then $B \not\subseteq C$.

*6. Prove that for any sets $A$, $B$, and $C$, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$, by finding a string of equivalences starting with $x \in A \setminus (B \cap C)$ and ending with $x \in (A \setminus B) \cup (A \setminus C)$ (See Example 3.4.4.)

*7. Use the methods of this chapter to prove that for any sets $A$ and $B$, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

*8. Prove that $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

*9. Prove that if $x$ and $y$ are odd integers, then $xy$ is odd.

10. Prove that for every integer $n$, $n^3$ is even iff $n$ is even.
11. Consider the following putative theorem:

**Theorem?** Suppose $m$ is an even integer and $n$ is an odd integer. Then $n^2 - m^2 = n + m$.

(a) What's wrong with the following proof of the theorem?

*Proof.* Since $m$ is even, we can choose some integer $k$ such that $m = 2k$. Similarly, since $n$ is odd we have $n = 2k + 1$. Therefore

$$n^2 - m^2 = (2k + 1)^2 - (2k)^2 = 4k^2 + 4k + 1 - 4k^2 = 4k + 1$$

$$= (2k + 1) + (2k) = n + m.$$\[\square\]

(b) Is the theorem correct? Justify your answer with either a proof or a counterexample.

*12. Prove that $\forall x \in \mathbb{R} [\exists y \in \mathbb{R} (x + y = x y) \iff x \neq 1]$.

13. Prove that $\exists x \in \mathbb{R} \forall x \in \mathbb{R}^* [\exists y \in \mathbb{R} (y - x = y/x) \iff x \neq 0]$.

14. Suppose $B$ is a set and $\mathcal{F}$ is a family of sets. Prove that $\bigcup\{A \setminus B \mid A \in \mathcal{F}\} \subseteq \bigcup(\mathcal{F} \setminus \mathcal{P}(B))$.

*15. Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets and every element of $\mathcal{F}$ is disjoint from some element of $\mathcal{G}$. Prove that $\bigcup \mathcal{F} \cap \bigcap \mathcal{G}$ are disjoint.

16. Prove that for any set $A$, $A = \bigcup \mathcal{P}(A)$.

*17. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.

(a) Prove that $\bigcup(\mathcal{F} \cap \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$.

(b) What's wrong with the following proof that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$?

*Proof.* Suppose $x \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$. This means that $x \in \bigcup \mathcal{F}$ and $x \in \bigcup \mathcal{G}$, so $\exists A \in \mathcal{F} (x \in A)$ and $\exists A \in \mathcal{G} (x \in A)$. Thus, we can choose a set $A$ such that $A \in \mathcal{F}$, $A \in \mathcal{G}$, and $x \in A$. Since $A \in \mathcal{F}$ and $A \in \mathcal{G}$, $A \in \mathcal{F} \cap \mathcal{G}$. Therefore $\exists A \in \mathcal{F} \cap \mathcal{G}$ such that $x \in A$, so $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$. Since $x$ was arbitrary, we can conclude that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$.\[\square\]

(c) Find an example of families of sets $\mathcal{F}$ and $\mathcal{G}$ for which $\bigcup(\mathcal{F} \cap \mathcal{G}) \neq (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$.

18. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$ if $\forall A \in \mathcal{F} \forall B \in \mathcal{G} (A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$).

19. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are disjoint if and only if for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, $A$ and $B$ are disjoint.

20. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.

(a) Prove that $(\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \setminus \mathcal{G})$.

(b) What's wrong with the following proof that $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$?
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Proof. Suppose \( x \in \cup(\mathcal{F} \setminus \mathcal{G}) \). Then we can choose some \( A \in \mathcal{F} \setminus \mathcal{G} \) such that \( x \in A \). Since \( A \in \mathcal{F} \setminus \mathcal{G} \), \( A \in \mathcal{F} \) and \( A \notin \mathcal{G} \). Since \( x \in A \) and \( A \in \mathcal{F} \), \( x \in \cup \mathcal{F} \). Since \( x \in A \) and \( A \notin \mathcal{G} \), \( x \notin \cup \mathcal{G} \). Therefore \( x \in (\cup \mathcal{F}) \setminus (\cup \mathcal{G}) \). \( \square \)

(c) Prove that \( \cup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\cup \mathcal{F}) \setminus (\cup \mathcal{G}) \) if \( \forall A \in (\mathcal{F} \setminus \mathcal{G}) \forall B \in \mathcal{G} (A \cap B = \emptyset) \).

(d) Find an example of families of sets \( \mathcal{F} \) and \( \mathcal{G} \) for which \( \cup(\mathcal{F} \setminus \mathcal{G}) \neq (\cup \mathcal{F}) \setminus (\cup \mathcal{G}) \).

*21. Suppose \( \mathcal{F} \) and \( \mathcal{G} \) are families of sets. Prove that if \( \cup \mathcal{F} \notin \cup \mathcal{G} \), then there is some \( A \in \mathcal{F} \) such that for all \( B \in \mathcal{G} \), \( A \notin B \).

22. Suppose \( B \) is a set, \( \{A_i \mid i \in I\} \) is an indexed family of sets, and \( I \neq \emptyset \).

(a) What proof strategies are used in the following proof that \( B \cap (\cup_{i \in I} A_i) = \cup_{i \in I} (B \cap A_i) \)?

Proof. Let \( x \) be arbitrary. Suppose \( x \in B \cap (\cup_{i \in I} A_i) \). Then \( x \in B \) and \( x \in \cup_{i \in I} A_i \), so we can choose some \( i_0 \in I \) such that \( x \in A_{i_0} \). Since \( x \in B \) and \( x \in A_{i_0} \), \( x \in B \cap A_{i_0} \). Therefore \( x \in \cup_{i \in I} (B \cap A_i) \).

Now suppose \( x \in \cup_{i \in I} (B \cap A_i) \). Then we can choose some \( i_0 \in I \) such that \( x \in B \cap A_{i_0} \). Therefore \( x \in B \) and \( x \in A_{i_0} \). Since \( x \in A_{i_0} \), \( x \in \cup_{i \in I} A_i \). Since \( x \in B \) and \( x \in \cup_{i \in I} A_i \), \( x \in B \cap (\cup_{i \in I} A_i) \).

Since \( x \) was arbitrary, we have shown that \( \forall x [x \in B \cap (\cup_{i \in I} A_i) \leftrightarrow x \in \cup_{i \in I} (B \cap A_i)] \), so \( B \cap (\cup_{i \in I} A_i) = \cup_{i \in I} (B \cap A_i) \). \( \square \)

(b) Prove that \( B \setminus (\cup_{i \in I} A_i) = \cap_{i \in I} (B \setminus A_i) \).

(c) Can you discover and prove a similar theorem about \( B \setminus (\cap_{i \in I} A_i) \)?

(Hint: Try to guess the theorem, and then try to prove it. If you can't finish the proof, it might be because your guess was wrong. Change your guess and try again.)

*23. Suppose \( \{A_i \mid i \in I\} \) and \( \{B_i \mid i \in I\} \) are indexed families of sets and \( I \neq \emptyset \).

(a) Prove that \( \cup_{i \in I} (A_i \setminus B_i) \subseteq (\cup_{i \in I} A_i) \setminus (\cap_{i \in I} B_i) \).

(b) Find an example for which \( \cup_{i \in I} (A_i \setminus B_i) \neq (\cap_{i \in I} A_i) \setminus (\cap_{i \in I} B_i) \).

24. Suppose \( \{A_i \mid i \in I\} \) and \( \{B_i \mid i \in I\} \) are indexed families of sets.

(a) Prove that \( \cup_{i \in I} (A_i \cap B_i) \subseteq (\cup_{i \in I} A_i) \cap (\cup_{i \in I} B_i) \).

(b) Find an example for which \( \cup_{i \in I} (A_i \cap B_i) \neq (\cup_{i \in I} A_i) \cap (\cup_{i \in I} B_i) \).

25. Prove that for all integers \( a \) and \( b \) there is an integer \( c \) such that \( a \mid c \) and \( b \mid c \).

26. (a) Prove that for every integer \( n \), \( 15 \mid n \) if and only if \( 3 \mid n \) and \( 5 \mid n \).

(b) Prove that it is not true that for every integer \( n \), \( 60 \mid n \) if and only if \( 6 \mid n \) and \( 10 \mid n \).