Proofs Involving Disjunctions

In fact, this rule is the one we used in our first example of deductive reasoning in Chapter 1!

Once again, we end this section with a proof for you to read without the benefit of a preliminary scratch work analysis.

**Theorem 3.5.5.** Suppose \( m \) and \( n \) are integers. If \( mn \) is even, then either \( m \) is even or \( n \) is even.

**Proof.** Suppose \( mn \) is even. Then we can choose an integer \( k \) such that \( mn = 2k \). If \( m \) is even then there is nothing more to prove, so suppose \( m \) is odd. Then \( m = 2j + 1 \) for some integer \( j \). Substituting this into the equation \( mn = 2k \), we get \( (2j + 1)n = 2k \), so \( 2jn + n = 2k \), and therefore \( n = 2k - 2jn = 2(k - jn) \). Since \( k - jn \) is an integer, it follows that \( n \) is even.

**Commentary.** The overall form of the proof is the following:

Suppose \( mn \) is even.

If \( m \) is even, then clearly either \( m \) is even or \( n \) is even. Now suppose \( m \) is not even. Then \( m \) is odd.

[Proof that \( n \) is even goes here.]

Therefore either \( m \) is even or \( n \) is even.

Therefore if \( mn \) is even then either \( m \) is even or \( n \) is even.

The assumptions that \( mn \) is even and \( m \) is odd lead, by existential instantiation, to the equations \( mn = 2k \) and \( m = 2j + 1 \). Although the proof doesn’t say so explicitly, you are expected to work out for yourself that in order to prove that \( n \) is even it suffices to find an integer \( c \) such that \( n = 2c \). Straightforward algebra leads to the equation \( n = 2(k - jn) \), so the choice \( c = k - jn \) works.

**Exercises**

\( \ast \)1. Suppose \( A, B, \) and \( C \) are sets. Prove that \( A \cap (B \cup C) \subseteq (A \cap B) \cup C \).

\( \ast \)2. Suppose \( A, B, \) and \( C \) are sets. Prove that \( (A \cup B) \setminus C \subseteq A \cup (B \setminus C) \).

\( \ast \)3. Suppose \( A \) and \( B \) are sets. Prove that \( A \setminus (A \setminus B) = A \cap B \).

\( \ast \)4. Suppose \( A \subseteq B \cap C \) and \( A \cup C \subseteq B \cup C \). Prove that \( A \subseteq B \).

\( \ast \)5. Recall from Section 1.4 that the symmetric difference of two sets \( A \) and \( B \) is the set \( A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B) \). Prove that if \( A \triangle B \subseteq A \) then \( B \subseteq A \).

\( \ast \)6. Suppose \( A, B, \) and \( C \) are sets. Prove that \( A \cup C \subseteq B \cup C \) iff \( A \setminus C \subseteq B \setminus C \).

\( \ast \ast \)7. Prove that for any sets \( A \) and \( B \), \( \mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B) \).
8. Prove that for any sets $A$ and $B$, if $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ then either $A \subseteq B$ or $B \subseteq A$.

9. Suppose $x$ and $y$ are real numbers and $x \neq 0$. Prove that $y + 1/x = 1 + y/x$ if either $x = 1$ or $y = 1$.

10. Prove that for every real number $x$, if $|x - 3| > 3$ then $x^2 > 6x$. (Hint: According to the definition of $|x - 3|$, if $x - 3 \geq 0$ then $|x - 3| = x - 3$, and if $x - 3 < 0$ then $|x - 3| = 3 - x$. The easiest way to use this fact is to break your proof into cases. Assume that $x - 3 \geq 0$ in case 1, and $x - 3 < 0$ in case 2.)

*11. Prove that for every real number $x$, $|2x - 6| > x$ iff $|x - 4| > 2$. (Hint: Read the hint for exercise 10.)

12. (a) Prove that for all real numbers $a$ and $b$, $|a| \leq b$ iff $-b \leq a \leq b$.

(b) Prove that for any real number $x$, $-|x| \leq x \leq |x|$. (Hint: Use part (a).)

(c) Prove that for all real numbers $x$ and $y$, $|x + y| \leq |x| + |y|$. (This is called the triangle inequality. One way to prove this is to combine parts (a) and (b), but you can also do it by considering a number of cases.)

13. Prove that for every integer $x$, $x^2 + x$ is even.

14. Prove that for every integer $x$, the remainder when $x^4$ is divided by 8 is either 0 or 1.

*15. Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets.

(a) Prove that $\cup(\mathcal{F} \cup \mathcal{G}) = (\cup \mathcal{F}) \cup (\cup \mathcal{G})$.

(b) Can you discover and prove a similar theorem about $\cap(\mathcal{F} \cup \mathcal{G})$?

16. Suppose $\mathcal{F}$ is a nonempty family of sets and $B$ is a set.

(a) Prove that $B \cup (\cup \mathcal{F}) = \cup(\mathcal{F} \cup \{B\})$.

(b) Prove that $B \cup (\cap \mathcal{F}) = \cap_{A \in \mathcal{F}}(B \cup A)$.

(c) Can you discover and prove a similar theorem about $B \cap (\cap \mathcal{F})$?

17. Suppose $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ are nonempty families of sets and for every $A \in \mathcal{F}$ and every $B \in \mathcal{G}$, $A \cup B \in \mathcal{H}$. Prove that $\cap \mathcal{H} \subseteq (\cap \mathcal{F}) \cup (\cap \mathcal{G})$.

18. Suppose $A$ and $B$ are sets. Prove that $\forall x(x \in A \triangle B \leftrightarrow (x \in A \leftrightarrow x \notin B))$.

*19. Suppose $A$, $B$, and $C$ are sets. Prove that $A \triangle B$ and $C$ are disjoint iff $A \cap C = B \cap C$.

20. Suppose $A$, $B$, and $C$ are sets. Prove that $A \triangle B \subseteq C$ iff $A \cup C = B \cup C$.

21. Suppose $A$, $B$, and $C$ are sets. Prove that $C \subseteq A \triangle B$ iff $C \subseteq A \cup B$ and $A \cap B \cap C = \emptyset$.

*22. Suppose $A$, $B$, and $C$ are sets.

(a) Prove that $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$.

(b) Prove that $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$. 
Suppose $A$, $B$, and $C$ are sets.
(a) Prove that $(A \cup B) \triangle C \subseteq (A \triangle C) \cup (B \triangle C)$.
(b) Find an example of sets $A$, $B$, and $C$ such that $(A \cup B) \triangle C \neq (A \triangle C) \cup (B \triangle C)$

Suppose $A$, $B$, and $C$ are sets.
(a) Prove that $(A \triangle C) \cap (B \triangle C) \subseteq (A \cap B) \triangle C$.
(b) Is it always true that $(A \cap B) \triangle C \subseteq (A \triangle C) \cap (B \triangle C)$? Give either a proof or a counterexample.

Suppose $A$, $B$, and $C$ are sets. Consider the sets $(A \setminus B) \triangle C$ and $(A \triangle C) \setminus (B \triangle C)$. Can you prove that either is a subset of the other? Justify your conclusions with either proofs or counterexamples.

Consider the following putative theorem.

**Theorem?** For every real number $x$, if $|x - 3| < 3$ then $0 < x < 6$.

Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

**Proof.** Let $x$ be an arbitrary real number, and suppose $|x - 3| < 3$. We consider two cases:

Case 1. $x - 3 \geq 0$. Then $|x - 3| = x - 3$. Plugging this into the assumption that $|x - 3| < 3$, we get $x - 3 < 3$, so clearly $x < 6$.

Case 2. $x - 3 < 0$. Then $|x - 3| = 3 - x$, so the assumption $|x - 3| < 3$ means that $3 - x < 3$. Therefore $3 < 3 + x$, so $0 < x$.

Since we have proven both $0 < x$ and $x < 6$, we can conclude that $0 < x < 6$.

Consider the following putative theorem.

**Theorem?** For any sets $A$, $B$, and $C$, if $A \setminus B \subseteq C$ and $A \nsubseteq C$ then $A \cap B \neq \emptyset$.

Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

**Proof.** Since $A \nsubseteq C$, we can choose some $x$ such that $x \in A$ and $x \notin C$. Since $x \notin C$ and $A \setminus B \subseteq C$, $x \notin A \setminus B$. Therefore either $x \notin A$ or $x \in B$. But we already know that $x \in A$, so it follows that $x \in B$. Since $x \in A$ and $x \in B$, $x \in A \cap B$. Therefore $A \cap B \neq \emptyset$.

Consider the following putative theorem.

**Theorem?** $\forall x \in \mathbb{R} \exists y \in \mathbb{R} (xy^2 \neq y - x)$.

Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?
Proof. Let $x$ be an arbitrary real number.

Case 1. $x = 0$. Let $y = 1$. Then $xy^2 = 0$ and $y - x = 1 - 0 = 1$, so $xy^2 
eq y - x$.

Case 2. $x 
eq 0$. Let $y = 0$. Then $xy^2 = 0$ and $y - x = -x 
eq 0$, so $xy^2 
eq y - x$.

Since these cases are exhaustive, we have shown that $\exists y \in \mathbb{R}(xy^2 \neq y - x)$. Since $x$ was arbitrary, this shows that $\forall x \in \mathbb{R}\exists y \in \mathbb{R}(xy^2 \neq y - x)$.

29. Prove that if $\forall x P(x) \rightarrow \exists x Q(x)$ then $\exists x(P(x) \rightarrow Q(x))$. (Hint: Remember that $P \rightarrow Q$ is equivalent to $\neg P \vee Q$).

*30. Consider the following putative theorem.

**Theorem?** Suppose $A$, $B$, and $C$ are sets and $A \subseteq B \cup C$. Then either $A \subseteq B$ or $A \subseteq C$.

Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

**Proof.** Let $x$ be an arbitrary element of $A$. Since $A \subseteq B \cup C$, it follows that either $x \in B$ or $x \in C$.

Case 1. $x \in B$. Since $x$ was an arbitrary element of $A$, it follows that $\forall x \in A(x \in B)$, which means that $A \subseteq B$.

Case 2. $x \in C$. Similarly, since $x$ was an arbitrary element of $A$, we can conclude that $A \subseteq C$.

Thus, either $A \subseteq B$ or $A \subseteq C$.

31. Prove $\exists x(P(x) \rightarrow \forall y P(y))$.

### 3.6. Existence and Uniqueness Proofs

In this section we consider proofs in which the goal has the form $\exists! x P(x)$. As we saw in Section 2.2, this can be thought of as an abbreviation for the formula $\exists x(P(x) \land \neg \exists y(P(y) \land y \neq x))$. According to the proof strategies discussed in previous sections, we could therefore prove this goal by finding a particular value of $x$ for which we could prove both $P(x)$ and $\neg \exists y(P(y) \land y \neq x)$. The last part of this proof would involve proving a negated statement, but we can reexpress it as an equivalent positive statement:

$\neg \exists y(P(y) \land y \neq x)$

is equivalent to $\forall y \neg (P(y) \land y \neq x)$ (quantifier negation law),

which is equivalent to $\forall y (\neg P(y) \lor y = x)$ (DeMorgan’s law),

which is equivalent to $\forall y (P(y) \rightarrow y = x)$ (conditional law),