one-to-one correspondence. Even without counting people or seats, we can tell that the number of people in the audience must be the same as the number of seats in the concert hall.

**Exercises**

1. Which of the functions in exercise 1 of Section 5.1 are one-to-one? Which are onto?
2. Which of the functions in exercise 2 of Section 5.1 are one-to-one? Which are onto?
3. Which of the functions in exercise 3 of Section 5.1 are one-to-one? Which are onto?
4. Which of the functions in exercise 4 of Section 5.1 are one-to-one? Which are onto?
5. Let $A = \mathbb{R} \setminus \{1\}$, and let $f : A \to A$ be defined as follows:

   $$f(x) = \frac{x + 1}{x - 1}.$$ 

   (a) Show that $f$ is one-to-one and onto.
   (b) Show that $f \circ f = i_A$.
6. Let $A = \mathcal{P}(\mathbb{R})$. Define $f : \mathbb{R} \to A$ by the formula $f(x) = \{y \in \mathbb{R} \mid y^2 < x\}$.
   (a) Find $f(2)$.
   (b) Is $f$ one-to-one? Is it onto?
7. Let $A = \mathcal{P}(\mathbb{R})$ and $B = \mathcal{P}(A)$. Define $f : B \to A$ by the formula $f(F) = \cup F$.
   (a) Find $f(\{\{1, 2\}, \{3, 4\}\})$.
   (b) Is $f$ one-to-one? Is it onto?
8. Suppose $f : A \to B$ and $g : B \to C$.
   (a) Prove that if $g \circ f$ is onto then $g$ is onto.
   (b) Prove that if $g \circ f$ is one-to-one then $f$ is one-to-one.
9. Suppose $f : A \to B$ and $g : B \to C$.
   (a) Prove that if $f$ is onto and $g$ is not one-to-one, then $g \circ f$ is not one-to-one.
   (b) Prove that if $f$ is not onto and $g$ is one-to-one, then $g \circ f$ is not onto.
10. Suppose $f : A \to B$ and $C \subseteq A$. In exercise 7 of Section 5.1 we defined $f \upharpoonright C$ (the restriction of $f$ to $C$), and you showed that $f \upharpoonright C : C \to B$.
    (a) Prove that if $f$ is one-to-one, then so is $f \upharpoonright C$.
    (b) Prove that if $f \upharpoonright C$ is onto, then so is $f$. 

(c) Give examples to show that the converses of parts (a) and (b) are not true.

11. Suppose \( f : A \to B \), and there is some \( b \in B \) such that \( \forall x \in A(f(x) = b) \). (Thus, \( f \) is a constant function.)
   (a) Prove that if \( A \) has more than one element then \( f \) is not one-to-one.
   (b) Prove that if \( B \) has more than one element then \( f \) is not onto.

12. Suppose \( f : A \to C \), \( g : B \to C \), and \( A \) and \( B \) are disjoint. In exercise 9(a) of Section 5.1 you proved that \( f \cup g : A \cup B \to C \). Now suppose in addition that \( f \) and \( g \) are both one-to-one. Prove that \( f \cup g \) is one-to-one iff \( \text{Ran}(f) \) and \( \text{Ran}(g) \) are disjoint.

13. Suppose \( R \) is a relation from \( A \) to \( B \), \( S \) is a relation from \( B \) to \( C \), \( \text{Ran}(R) = \text{Dom}(S) = B \), and \( S \circ R : A \to C \). In exercise 10(a) of Section 5.1 you proved that \( S : B \to C \). Now prove that if \( S \) is one-to-one then \( R : A \to B \).

*14. Suppose \( f : A \to B \) and \( R \) is a relation on \( A \). As in exercise 12 of Section 5.1, define a relation \( S \) on \( B \) as follows:

\[
S = \{(x, y) \in B \times B \mid \exists u \in A \exists v \in A(f(u) = x \land f(v) = y \land (u, v) \in R)\}.
\]

   (a) Prove that if \( R \) is reflexive and \( f \) is onto then \( S \) is reflexive.
   (b) Prove that if \( R \) is transitive and \( f \) is one-to-one then \( S \) is transitive.

15. Suppose \( R \) is an equivalence relation on \( A \), and let \( g : A \to A/R \) be defined by the formula \( g(x) = [x]_R \), as in exercise 17 in Section 5.1.
   (a) Show that \( g \) is onto.
   (b) Show that \( g \) is one-to-one iff \( R = i_A \).

16. Suppose \( f : A \to B \), \( R \) is an equivalence relation on \( A \), and \( f \) is compatible with \( R \). (See exercise 18 of Section 5.1 for the definition of compatible.) In exercise 18(a) of Section 5.1 you proved that there is a unique function \( h : A/R \to B \) such that for all \( x \in A \), \( h([x]_R) = f(x) \). Now prove that \( h \) is one-to-one iff \( \forall x \in A \forall y \in A(f(x) = f(y) \implies xRy) \).

*17. Suppose \( A, B, \) and \( C \) are sets and \( f : A \to B \).
   (a) Prove that if \( f \) is onto, \( g : B \to C \), \( h : B \to C \), and \( g \circ f = h \circ f \), then \( g = h \).
   (b) Suppose that \( C \) has at least two elements, and for all functions \( g \) and \( h \) from \( B \) to \( C \), if \( g \circ f = h \circ f \) then \( g = h \). Prove that \( f \) is onto.

18. Suppose \( A, B, \) and \( C \) are sets and \( f : B \to C \).
   (a) Prove that if \( f \) is one-to-one, \( g : A \to B \), \( h : A \to B \), and \( f \circ g = f \circ h \), then \( g = h \).
   (b) Suppose that \( A \neq \emptyset \), and for all functions \( g \) and \( h \) from \( A \) to \( B \), if \( f \circ g = f \circ h \) then \( g = h \). Prove that \( f \) is one-to-one.