Functions

What went wrong? We know that if $f^{-1}$ is a function from $B$ to $A$, then for any $x \in B$, $f^{-1}(x)$ must be the unique solution for $y$ in the equation $f(y) = x$. Applying the definition of $f$ gives us $y^2 = x$, so $y = \pm \sqrt{x}$. Thus, there is not a unique solution for $y$ in the equation $f(y) = x$, there are two solutions. For example, when $x = 9$ we get $y = \pm 3$. In other words, $f(3) = f(-3) = 9$. But this means that $f$ is not one-to-one! Thus, $f^{-1}$ is not a function from $B$ to $A$.

Functions that undo each other come up often in mathematics. For example, if you are familiar with logarithms, then you will recognize the formulas $10^{\log x} = x$ and $\log 10^x = x$. (We are using base 10 logarithms here.) We can rephrase these formulas in the language of this section by defining functions $f : \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}$ as follows:

$$f(x) = 10^x, \quad g(x) = \log x.$$  

Then for any $x \in \mathbb{R}$ we have $g(f(x)) = \log 10^x = x$, and for any $x \in \mathbb{R}^+$, $f(g(x)) = 10^{\log x} = x$. Thus, $g \circ f = i_\mathbb{R}$ and $f \circ g = i_{\mathbb{R}^+}$, so $g = f^{-1}$. In other words, the logarithm function is the inverse of the “raise 10 to the power” function.

We saw another example of functions that undo each other in Section 4.6. Suppose $A$ is any set, let $\mathcal{E}$ be the set of all equivalence relations on $A$, and let $\mathcal{P}$ be the set of all partitions of $A$. Define a function $f : \mathcal{E} \to \mathcal{P}$ by the formula $f(\mathcal{E}) = A / \mathcal{R}$, and define another function $g : \mathcal{P} \to \mathcal{E}$ by the formula

$$g(\mathcal{F}) = \text{the equivalence relation determined by } \mathcal{F} = \bigcup_{x \in \mathcal{F}} X \times X.$$  

You should verify that the proof of Theorem 4.6.6 shows that $f \circ g = i_\mathcal{P}$, and exercise 9 in Section 4.6 shows that $g \circ f = i_\mathcal{E}$. Thus, $f$ is one-to-one and onto, and $g = f^{-1}$. One interesting consequence of this is that if $A$ has a finite number of elements, then we can say that the number of equivalence relations on $A$ is exactly the same as the number of partitions of $A$, even though we don’t know what this number is.

Exercises

*1. Let $R$ be the function defined in exercise 2(c) of Section 5.1. In exercise 2 of Section 5.2, you showed that $R$ is one-to-one and onto, so $R^{-1} : P \to P$. If $p \in P$, what is $R^{-1}(p)$?
2. Let \( F \) be the function defined in exercise 4(b) of Section 5.1. In exercise 4 of Section 5.2, you showed that \( F \) is one-to-one and onto, so \( F^{-1} : B \to B \). If \( X \in B \), what is \( F^{-1}(X) \)?

*3. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by the formula

\[
    f(x) = \frac{2x + 5}{3}.
\]

Show that \( f \) is one-to-one and onto, and find a formula for \( f^{-1}(x) \). (You may want to imitate the method used in the example after Theorem 5.3.2, or in Example 5.3.6.)

4. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by the formula \( f(x) = 2x^3 - 3 \). Show that \( f \) is one-to-one and onto, and find a formula for \( f^{-1}(x) \).

*5. Let \( f : \mathbb{R} \to \mathbb{R}^+ \) be defined by the formula \( f(x) = 10^{2-x} \). Show that \( f \) is one-to-one and onto, and find a formula for \( f^{-1}(x) \).

6. Let \( A = \mathbb{R} \setminus \{2\} \), and let \( f \) be the function with domain \( A \) defined by the formula

\[
    f(x) = \frac{3x}{x - 2}.
\]

(a) Show that \( f \) is a one-to-one, onto function from \( A \) to \( B \) for some set \( B \subseteq \mathbb{R} \). What is the set \( B \)?

(b) Find a formula for \( f^{-1}(x) \).

7. In the example after Theorem 5.3.4, we had \( f(x) = \frac{x + 7}{5} \) and found that \( f^{-1}(x) = 5x - 7 \). Let \( f_1 \) and \( f_2 \) be functions from \( \mathbb{R} \) to \( \mathbb{R} \) defined by the formulas

\[
    f_1(x) = x + 7, \quad f_2(x) = \frac{x}{5}.
\]

(a) Show that \( f = f_2 \circ f_1 \).

(b) According to part 5 of Theorem 4.2.5, we must have \( f^{-1} = (f_2 \circ f_1)^{-1} = (f_1)^{-1} \circ (f_2)^{-1} \). Verify that this is true by computing \( (f_1)^{-1} \circ (f_2)^{-1} \) directly.

8. (a) Prove the second half of Theorem 5.3.2 by imitating the proof of the first half.

(b) Give an alternative proof of the second half of Theorem 5.3.2 by applying the first half to \( f^{-1} \).

*9. Prove part 2 of Theorem 5.3.3.

10. Use the following strategy to give an alternative proof of Theorem 5.3.5:

Let \((b, a)\) be an arbitrary element of \( B \times A \). Assume \((b, a) \in g \) and prove \((b, a) \in f^{-1} \). Then assume \((b, a) \in f^{-1} \) and prove \((b, a) \in g \).