The base case $n = 5$ has already been checked in the table. For the induction step, we let $n \geq 5$ be arbitrary, assume $2^n > n^2$, and try to prove that $2^{n+1} > (n + 1)^2$. How can we relate the inductive hypothesis to the goal? Perhaps the simplest relationship involves the left sides of the two inequalities: $2^{n+1} = 2 \cdot 2^n$. Thus, multiplying both sides of the inductive hypothesis $2^n > n^2$ by 2, we can conclude that $2^{n+1} > 2n^2$. Now compare this inequality to the goal, $2^{n+1} > (n + 1)^2$. If we could prove that $2n^2 \geq (n + 1)^2$, then the goal would follow easily. So let’s forget about the original goal and see if we can prove that $2n^2 \geq (n + 1)^2$.

Multiplying out the right side of the new goal we see that we must prove that $2n^2 \geq n^2 + 2n + 1$, or in other words $n^2 \geq 2n + 1$. This isn’t hard to prove: Since we’ve assumed that $n \geq 5$, it follows that $n^2 \geq 5n = 2n + 3n > 2n + 1$.

Solution

**Theorem.** For every natural number $n \geq 5$, $2^n > n^2$.

**Proof.** By mathematical induction.

Base case: When $n = 5$ we have $2^5 = 32 > 25 = n^2$.

Induction step: Let $n \geq 5$ be arbitrary, and suppose that $2^n > n^2$. Then

\[
2^{n+1} = 2 \cdot 2^n \\
> 2n^2 \quad \text{(by inductive hypothesis)} \\
= n^2 + n^2 \\
\geq n^2 + 5n \quad \text{(since } n \geq 5) \\
= n^2 + 2n + 3n \\
> n^2 + 2n + 1 = (n + 1)^2. \tag*{\square}
\]

**Exercises**

1. Prove that for all $n \in \mathbb{N}$, $0 + 1 + 2 + \cdots + n = n(n + 1)/2$.
2. Prove that for all $n \in \mathbb{N}$, $0^2 + 1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$.
3. Prove that for all $n \in \mathbb{N}$, $0^3 + 1^3 + 2^3 + \cdots + n^3 = [n(n + 1)/2]^2$.
4. Find a formula for $1 + 3 + 5 + \cdots + (2n - 1)$, for $n \geq 1$, and prove that your formula is correct. (Hint: First try some particular values of $n$ and look for a pattern.)
5. Prove that for all $n \in \mathbb{N}$, $0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n + 1) = n(n + 1)(n + 2)/3$.
6. Find a formula for $0 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n + 1)(n + 2)$, for $n \in \mathbb{N}$, and prove that your formula is correct. (Hint: Compare this exercise to exercises 1 and 5, and try to guess the formula.)
7. Find a formula for $3^0 + 3^1 + 3^2 + \cdots + 3^6$, for $n \geq 0$, and prove that your formula is correct. (Hint: Try to guess the formula, basing your guess on Example 6.1.1. Then try out some values of $n$ and adjust your guess if necessary.)

8. Prove that for all $n \geq 1$,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$$

9. (a) Prove that all $n \in \mathbb{N}$, $2 \mid (n^2 + n)$.

(b) Prove that for all $n \in \mathbb{N}$, $6 \mid (n^3 - n)$.

*10. Prove that for all $n \in \mathbb{N}$, $64 \mid (9^n - 8n - 1)$.

11. Prove that for all $n \in \mathbb{N}$, $9 \mid (4^n + 6n - 1)$.

12. Prove that for all integers $a$ and $b$ and all $n \in \mathbb{N}$, $(a - b) \mid (a^n - b^n)$.

(Hint: Let $a$ and $b$ be arbitrary integers and then prove by induction that $\forall n \in \mathbb{N}[(a - b) \mid (a^n - b^n)]$. For the induction step, you must relate $a^{n+1} - b^{n+1}$ to $a^n - b^n$. You might find it useful to start by completing the following equation: $a^{n+1} - b^{n+1} = a(a^n - b^n) + ?$.)

13. Prove that for all integers $a$ and $b$ and all $n \in \mathbb{N}$, $(a + b) \mid (a^{2n+1} + b^{2n+1})$.

*14. Prove that for all $n \geq 10, 2^n > n^3$.

15. Prove that for all $n \in \mathbb{N}$, either $n \equiv 0 \pmod{3}$ or $n \equiv 1 \pmod{3}$ or $n \equiv 2 \pmod{3}$. (Recall that this notation was introduced in Definition 4.6.9.)

16. Prove that for all $n \geq 1$, $2 \cdot 2^1 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + (n + 1)2^n = n2^{n+1}$.

17. (a) What’s wrong with the following proof that for all $n \in \mathbb{N}$, $1 \cdot 3^0 + 3 \cdot 3^1 + 5 \cdot 3^2 + \cdots + (2n+1)3^n = n3^{n+1}$?

**Proof**. We use mathematical induction. Let $n$ be an arbitrary natural number, and suppose that $1 \cdot 3^0 + 3 \cdot 3^1 + 5 \cdot 3^2 + \cdots + (2n+1)3^n = n3^{n+1}$. Then

$$1 \cdot 3^0 + 3 \cdot 3^1 + 5 \cdot 3^2 + \cdots + (2n+1)3^n + (2n+3)3^{n+1}$$

$$= n3^{n+1} + (2n+3)3^{n+1}$$

$$= (3n + 3)3^{n+1}$$

$$= (n+1)3^{n+2},$$

as required.

(b) Find a formula for $1 \cdot 3^0 + 3 \cdot 3^1 + 5 \cdot 3^2 + \cdots + (2n+1)3^n$, and prove that your formula is correct.