Functions

What went wrong? We know that if $f^{-1}$ is a function from $B$ to $A$, then for any $x \in B$, $f^{-1}(x)$ must be the unique solution for $y$ in the equation $f(y) = x$. Applying the definition of $f$ gives us $y^2 = x$, so $y = \pm \sqrt{x}$. Thus, there is not a unique solution for $y$ in the equation $f(y) = x$, there are two solutions. For example, when $x = 9$ we get $y = \pm 3$. In other words, $f(3) = f(-3) = 9$. But this means that $f$ is not one-to-one! Thus, $f^{-1}$ is not a function from $B$ to $A$.

Functions that undo each other come up often in mathematics. For example, if you are familiar with logarithms, then you will recognize the formulas $10^{\log x} = x$ and $\log 10^x = x$. (We are using base 10 logarithms here.) We can rephrase these formulas in the language of this section by defining functions $f : \mathbb{R} \rightarrow \mathbb{R}^+$ and $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ as follows:

$$f(x) = 10^x, \quad g(x) = \log x.$$ 

Then for any $x \in \mathbb{R}$ we have $g(f(x)) = \log 10^x = x$, and for any $x \in \mathbb{R}^+$, $f(g(x)) = 10^{\log x} = x$. Thus, $g \circ f = i_\mathbb{R}$ and $f \circ g = i_{\mathbb{R}^+}$, so $g = f^{-1}$. In other words, the logarithm function is the inverse of the “raise 10 to the power” function.

We saw another example of functions that undo each other in Section 4.6. Suppose $A$ is any set, let $\mathcal{E}$ be the set of all equivalence relations on $A$, and let $\mathcal{P}$ be the set of all partitions of $A$. Define a function $f : \mathcal{E} \rightarrow \mathcal{P}$ by the formula $f(R) = A / R$, and define another function $g : \mathcal{P} \rightarrow \mathcal{E}$ by the formula

$$g(\mathcal{F}) = \text{the equivalence relation determined by } \mathcal{F}$$
$$= \bigcup_{X \in \mathcal{F}} (X \times X).$$

You should verify that the proof of Theorem 4.6.6 shows that $f \circ g = i_\mathcal{P}$, and exercise 9 in Section 4.6 shows that $g \circ f = i_\mathcal{E}$. Thus, $f$ is one-to-one and onto, and $g = f^{-1}$. One interesting consequence of this is that if $A$ has a finite number of elements, then we can say that the number of equivalence relations on $A$ is exactly the same as the number of partitions of $A$, even though we don’t know what this number is.

Exercises

*1. Let $R$ be the function defined in exercise 2(c) of Section 5.1. In exercise 2 of Section 5.2, you showed that $R$ is one-to-one and onto, so $R^{-1} : P \rightarrow P$. If $p \in P$, what is $R^{-1}(p)$?
2. Let \( F \) be the function defined in exercise 4(b) of Section 5.1. In exercise 4 of Section 5.2, you showed that \( F \) is one-to-one and onto, so \( F^{-1} : B \rightarrow B \). If \( X \in B \), what is \( F^{-1}(X) \)?

*3. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by the formula

\[
f(x) = \frac{2x + 5}{3}.
\]

Show that \( f \) is one-to-one and onto, and find a formula for \( f^{-1}(x) \). (You may want to imitate the method used in the example after Theorem 5.3.2, or in Example 5.3.6.)

4. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by the formula \( f(x) = 2x^3 - 3 \). Show that \( f \) is one-to-one and onto, and find a formula for \( f^{-1}(x) \).

*5. Let \( f : \mathbb{R} \rightarrow \mathbb{R}^+ \) be defined by the formula \( f(x) = 10^{2-x} \). Show that \( f \) is one-to-one and onto, and find a formula for \( f^{-1}(x) \).

6. Let \( A = \mathbb{R} \setminus \{2\} \), and let \( f \) be the function with domain \( A \) defined by the formula

\[
f(x) = \frac{3x}{x - 2}.
\]

(a) Show that \( f \) is a one-to-one, onto function from \( A \) to \( B \) for some set \( B \subseteq \mathbb{R} \). What is the set \( B \)?

(b) Find a formula for \( f^{-1}(x) \).

7. In the example after Theorem 5.3.4, we had \( f(x) = \frac{x+2}{5} \) and found that \( f^{-1}(x) = 5x - 7 \). Let \( f_1 \) and \( f_2 \) be functions from \( \mathbb{R} \) to \( \mathbb{R} \) defined by the formulas

\[
f_1(x) = x + 7, \quad f_2(x) = \frac{x}{5}.
\]

(a) Show that \( f = f_2 \circ f_1 \).

(b) According to part 5 of Theorem 4.2.5, we must have \( f^{-1} = (f_2 \circ f_1)^{-1} = (f_1)^{-1} \circ (f_2)^{-1} \). Verify that this is true by computing \((f_1)^{-1} \circ (f_2)^{-1} \) directly.

8. (a) Prove the second half of Theorem 5.3.2 by imitating the proof of the first half.

(b) Give an alternative proof of the second half of Theorem 5.3.2 by applying the first half to \( f^{-1} \).

*9. Prove part 2 of Theorem 5.3.3.

10. Use the following strategy to give an alternative proof of Theorem 5.3.5:

Let \((b, a)\) be an arbitrary element of \( B \times A \). Assume \((b, a) \in g \) and prove \((b, a) \in f^{-1} \). Then assume \((b, a) \in f^{-1} \) and prove \((b, a) \in g \).
Functions

11. Suppose \( f : A \rightarrow B \) and \( g : B \rightarrow A \).
   (a) Prove that if \( f \) is one-to-one and \( f \circ g = i_B \), then \( g = f^{-1} \).
   (b) Prove that if \( f \) is onto and \( g \circ f = i_A \), then \( g = f^{-1} \).
   (c) Prove that if \( f \circ g = i_B \) but \( g \circ f \neq i_A \), then \( f \) is onto but not
       one-to-one, and \( g \) is one-to-one but not onto.

12. Suppose \( f : A \rightarrow B \) and \( f \) is one-to-one. Prove that there is some set
    \( B' \subseteq B \) such that \( f^{-1} : B' \rightarrow A \).

13. Suppose \( f : A \rightarrow B \) and \( f \) is onto. Let \( R = \{ (x, y) \in A \times A \mid f(x) = f(y) \} \). By exercise 17(a)
    of Section 5.1, \( R \) is an equivalence relation on \( A \).
   (a) Prove that there is a function \( h : A/R \rightarrow B \) such that for all \( x \in A \),
       \( h([x]_R) = f(x) \). (Hint: See exercise 18 of Section 5.1.)
   (b) Prove that \( h \) is one-to-one and onto. (Hint: See exercise 16 of Section
       5.2.)
   (c) It follows from part (b) that \( h^{-1} : B \rightarrow A/R \). Prove that for all \( b \in B \),
       \( h^{-1}(b) = \{ x \in A \mid f(x) = b \} \).
   (d) Suppose \( g : B \rightarrow A \). Prove that \( f \circ g = i_B \) if and only if \( \forall b \in B \),
       \( g(b) \in h(b) \).

14. Suppose \( f : A \rightarrow B, g : B \rightarrow A \), and \( f \circ g = i_B \). Let \( A' = \text{Ran}(g) \subseteq A \).
    (a) Prove that for all \( x \in A' \), \( (g \circ f)(x) = x \).
    (b) Prove that \( f \upharpoonright A' \) is a one-to-one, onto function from \( A' \) to \( B \) and
       \( g = (f \upharpoonright A')^{-1} \). (See exercise 7 of Section 5.1 for the meaning of
       the notation used here.)

15. Let \( B = \{ x \in \mathbb{R} \mid x \geq 0 \} \). Let \( f : \mathbb{R} \rightarrow B \) and \( g : B \rightarrow \mathbb{R} \) be defined by
    the formulas \( f(x) = x^2 \) and \( g(x) = \sqrt{x} \). As we saw in part 2 of Example
    5.3.6, \( g \neq f^{-1} \). Show that \( g = (f \upharpoonright B)^{-1} \). (Hint: See exercise 14.)

16. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by the formula \( f(x) = 4x - x^2 \). Let \( B = \text{Ran}(f) \).
    (a) Find \( B \).
    (b) Find a set \( A \subseteq \mathbb{R} \) such that \( f \upharpoonright A \) is a one-to-one, onto
        function from \( A \) to \( B \), and find a formula for \((f \upharpoonright A)^{-1}\). (Hint: See exercise 14.)

17. Let \( A \) be any set. Let \( \mathcal{P} \) be the set of all partial orders on \( A \), and let \( \mathcal{S} \) be the
    set of all strict partial orders on \( A \). In exercises 4 and 5 of Section 4.5 you
    showed that if \( R \in \mathcal{P} \) then \( R \setminus i_A \in \mathcal{S} \), and if \( R \in \mathcal{S} \) then \( R \cup i_A \in \mathcal{P} \).
    (Recall that we showed in the proof of Theorem 4.5.2 that \( R \cup i_A \) is the
    reflexive closure of \( R \).) Let \( f : \mathcal{P} \rightarrow \mathcal{S} \) and \( g : \mathcal{S} \rightarrow \mathcal{P} \) be defined by
    the formulas
    \[
    f(R) = R \setminus i_A, \quad g(R) = R \cup i_A.
    \]
    Show that \( f \) is one-to-one and onto, and \( g = f^{-1} \).
18. Suppose \( A \) is a set, and let \( \mathcal{F} = \{ f \mid f : A \to A \} \) and \( \mathcal{P} = \{ f \in \mathcal{F} \mid f \) is one-to-one and onto\}. Define a relation \( R \) on \( \mathcal{F} \) as follows:

\[
R = \{ (f, g) \in \mathcal{F} \times \mathcal{F} \mid \exists h \in \mathcal{P}(f = h^{-1} \circ g \circ h) \}.
\]

(a) Prove that \( R \) is an equivalence relation.
(b) Prove that if \( f R g \) then \( (f \circ f)R(g \circ g) \).
(c) For any \( f \in \mathcal{F} \) and \( a \in A \), if \( f(a) = a \) then we say that \( a \) is a fixed point of \( f \). Prove that if \( f \) has a fixed point and \( f R g \), then \( g \) also has a fixed point.

5.4. Images and Inverse Images: A Research Project

Suppose \( f : A \to B \). We have already seen that we can think of \( f \) as matching each element of \( A \) with exactly one element of \( B \). In this section we will see that \( f \) can also be thought of as matching subsets of \( A \) with subsets of \( B \) and vice-versa.

**Definition 5.4.1.** Suppose \( f : A \to B \) and \( X \subseteq A \). Then the image of \( X \) under \( f \) is the set \( f(X) \) defined as follows:

\[
f(X) = \{ f(x) \mid x \in X \}
= \{ b \in B \mid \exists x \in X (f(x) = b) \}.
\]

(Note that the image of the whole domain \( A \) under \( f \) is \( \{ f(a) \mid a \in A \} \), and as we saw in Section 5.1 this is the same as the range of \( f \).)

If \( Y \subseteq B \), then the inverse image of \( Y \) under \( f \) is the set \( f^{-1}(Y) \) defined as follows:

\[
f^{-1}(Y) = \{ a \in A \mid f(a) \in Y \}.
\]

Note that the function \( f \) in Definition 5.4.1 may fail to be one-to-one or onto, and as a result \( f^{-1} \) may not be a function from \( B \) to \( A \), and for \( y \in B \), the notation \( f^{-1}(y) \) may be meaningless. However, even in this case Definition 5.4.1 still assigns a meaning to the notation \( f^{-1}(Y) \) for \( Y \subseteq B \). If you find this surprising, look again at the definition of \( f^{-1}(Y) \), and notice that it does not treat \( f^{-1} \) as a function. The definition refers only to the results of applying \( f \) to elements of \( A \), not the results of applying \( f^{-1} \) to elements of \( B \).

For example, let \( L \) be the function defined in part 3 of Example 5.1.2, which assigns to each city the country in which that city is located. As in Example 5.1.2, let \( C \) be the set of all cities and \( N \) the set of all countries. If \( B \) is the